MATH3705D — Test 1: 14:35–15:25, Feb 2

Surname ______________ **First Name** ______________ **Student #** __________

Total points: 15. No partial marks for Questions 1-4.
Closed book! Non-programmer calculators are allowed!

[**2**] 1. Calculate $L\{(t^2-t)\sqrt{t}\}$.

(A) $\frac{(15-4s^2)\sqrt{\pi}}{8s^{3.5}}$ (B) $\frac{(15-6s)\sqrt{\pi}}{8s^{3.5}}$ (C) $\frac{(35-8s^2)\sqrt{\pi}}{8s^{3.5}}$ (D) $\frac{(35-2s^2)\sqrt{\pi}}{8s^{3.5}}$ (E) $\frac{(15-2s^2)\sqrt{\pi}}{8s^{3.5}}$

Solution: (B)

$$L\{(t^2-t)\sqrt{t}\} = L\{t^{2.5}\} - L\{t^{1.5}\} = \frac{\Gamma(3.5)}{s^{3.5}} - \frac{\Gamma(2.5)}{s^{2.5}} = \frac{15\sqrt{\pi}}{8s^{3.5}} - \frac{3\sqrt{\pi}}{4s^{2.5}} = \frac{(15-6s)\sqrt{\pi}}{8s^{3.5}}.$$

[**2**] 2. $L\{t\cos(3t)\} =$

(A) $\frac{s^2-9}{(s^2+9)^2}$ (B) $\frac{9-s^2}{(s^2+9)^2}$ (C) $\frac{2\sin s}{(s^2+9)^2}$ (D) $-\frac{2e^s}{(s^2+3)^2}$ (E) $\frac{6s}{(s^2+9)^2}$

Solution: (A)

$L\{\cos 3t\} = \frac{s}{s^2+9}, \Rightarrow L\{t\cos 3t\} = -\left(\frac{s}{s^2+9}\right)' = \frac{s^2-9}{(s^2+9)^2}$.

[**2**] 3. Find $L\{f(t)\}$, where $f(t)$ satisfies $2f(t) + 2\int_0^t f(x)dx = t^3$.

(A) $\frac{3}{2s-1}$ (B) $\frac{3}{1-2s}$ (C) $\frac{1}{s^2(s+1)}$ (D) $\frac{6}{2s^2+s}$ (E) $\frac{3}{s^3(s+1)}$

Solution: (E)

Taking the Laplace transform we get

$$2F(s) + 2\frac{F(s)}{s} = \frac{3!}{s^4}, \Rightarrow F(s) = \frac{3}{s^3(s+1)}.$$

[**2**] 4. Find $\frac{1}{u(t-3)}L^{-1}\left\{\frac{e^{-3s}}{s+3}\right\}$.

(A) e^{-3t} (B) e^{t+3} (C) e^{-3t+9} (D) e^{-3t-9} (E) e^{-3t+3}

Solution: (C). By the Second Shift Theorem,

$$L^{-1}\left\{\frac{e^{-3s}}{s+3}\right\} = u(t-3)L^{-1}\left\{\frac{1}{s+3}\right\}_{t-3} = u(t-3)e^{-3(t-3)}.$$

[4] 5. Using Laplace transform solve $y''(t) + 2y'(t) - 8y(t) = 0$ with $y(0) = 0$, $y'(0) = 12$.

Solution: Applying Laplace transform to the two sides, we get the equation

$$s^2Y + 2sY - 8Y - 12 = 0, \Rightarrow Y(s) = \frac{12}{(s+4)(s-2)}. \qquad (2points)$$

By partial fraction method,

$$Y(s) = \frac{12}{(s+4)(s-2)} = \frac{2}{s-2} - \frac{2}{s+4}, \qquad (1point)$$

Thus

$$y(t) = 2e^{2t} - 2e^{-4t}. \qquad (1point)$$

[3] 6. Let $G(s) = \frac{s+5}{s^2+8s+17}$. Find $L^{-1}\{G(s)\}$.

Solution: We have

$$G(s) = \frac{s+5}{s^2+8s+17} = \frac{s+4}{(s+4)^2+1} + \frac{1}{(s+4)^2+1}. \qquad (2points)$$

Thus

$$g(t) = e^{-4t}\cos(t) + e^{-4t}\sin(t). \qquad (1point)$$

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\left\{t^p\right\} = \frac{\Gamma(p+1)}{s^{p+1}},\ p > -1$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha}F\left(\frac{s}{\alpha}\right), \quad \alpha > 0$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad s > a \geq 0$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n}F(s), \quad n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x)\,dx\right\} = F(s)G(s), \text{ where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt \text{ if } f \text{ is periodic with period } T$$