Solution - Version 2

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Discrete Mathematics for Computing - MAT 1348 B

Midterm Examination

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Instructions:

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators (without graphing or programming function) are allowed, but not needed.
- The exam consists of 11 questions on 10 pages. Page 10 is for additional work. Please do not detach it.
- Questions 1-6 are multiple-choice. You must enter the letter corresponding to each correct answer in the table preceding Question 1. No partial marks will be given for other work.
- Questions 7-11 are long-answer. You must clearly show all relevant steps in your solution to receive full marks. Clearly indicate the final answer.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work, you may use the back pages. Do not use scrap paper of your own.
- Use proper mathematical notation and terminology.
- If you require clarification, raise your hand.
- Good luck!

Last name:________________________

First name:________________________

Student number:________________________

Question	1 − 6	7	8	9	10	11	Total
Max	12	3	4	5	4	5	33
Marks							

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Questions 1–6 are multiple choice. Enter the **letter** corresponding to each correct answer in the appropriate box below.

Question	1	2	3	4	5	6
Answer	D	B	F	B	F	B

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1. The truth table of a compound proposition p with atomic propositions A, B, and C is as follows:

A	B	C	p
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

Only one of the following propositions is a **disjunctive normal form** of p — which one?

A. $(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge \neg C)$
B. $\neg A \vee \neg B \vee \neg C$
C. $(A \vee \neg B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee \neg B \vee \neg C)$
D. $(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge \neg C)$
E. $\neg A \wedge \neg B \wedge \neg C$
F. $(A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$

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2. Let S and T be finite sets with $|S| = 2$ and $|T| = 5$. What is the **cardinality of the power set of** $S \times T$?

A. 512 **B.** 1024 **C.** 4 **D.** 10 **E.** 16 **F.** 32

$|S \times T| = |S| \cdot |T| = 2 \cdot 5 = 10$

$|\mathcal{P}(S \times T)| = 2^{10} = 1024$

3. Which of the following statements are **true**?

F (i) The compound propositions $\neg((a \to b) \to c)$ and $a \wedge b \wedge \neg c$ are logically equivalent.

F (ii) The compound proposition $(a \to b) \to b$ is a tautology.

T (iii) If the set of premises of an argument is inconsistent, then the argument is valid.

T (iv) If X is false, Y is true, and Z is false, then $X \wedge Y \to Z$ is true.

A. only (iii) **B.** only (iv) **C.** only (i) **D.** only (ii) **E.** (ii) and (iv)

F. (iii) and (iv)

i) $\neg((a\to b)\to c) \equiv \neg(\neg(a\to b) \vee c) \equiv \neg(\neg(\neg a \vee b) \vee c) \equiv \neg(a \wedge \neg b \vee c) \equiv \neg a \vee b \wedge \neg c$

ii)

a	b	$a\to b$	$(a\to b)\to b$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

iii) If $\{H_1, \dots, H_n\}$ is inconsistent, then $H_1 \wedge \dots \wedge H_n$ is F and so $H_1 \wedge \dots \wedge H_n \to C$ is T

iv) Since $X \wedge Y$ is F, then $X \wedge Y \to Z$ is T

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4. On the Island of Knights and Knaves you meet two inhabitants A and B. Person B says: "A is a knave only if I am a knave." Which of the following statements is **true**?

(i) A is a knave and B is a knight.

(ii) A is a knight and B is a knave.

(iii) A and B are both knights.

(iv) A and B are both knaves.

(v) A is a knight but it is impossible to determine what B is.

(vi) B is a knight but it is impossible to determine what A is.

A. (v) **B.** (iii) **C.** (i) **D.** (vi) **E.** (iv) **F.** (ii)

p: "A is a knight" ; q: "B is a knight".

B says: $\neg p \to \neg q$

p	q	$\neg p \to \neg q$
T	T	T ←
T	F	T
F	T	F
F	F	T

q and $\neg p \to \neg q$ need to have the same truth value

So, p: T
q: T
∴ A and B are both knights.

5. Let $S = \{1, \{2\}, \{1,2\}, \emptyset\}$. Which of the following statements are **true**?

F (i) $\{\emptyset\} \in S$ — $\{0\}$ is not an element of S.

F (ii) $\{\{1\}, \emptyset\} \subseteq S$ — $\{1\}$ is not an element of S.

F (iii) $\{1, \{2\}\} \in S$ — $\{1, \{2\}\}$ is not an element of S

T (iv) $\{1, \{1,2\}\} \subseteq S$ — $\{1\}$ and $\{1,2\}$ are both elements of S

F (v) $\{1,2\} \subseteq S$ — 2 is not an element of S

F (vi) The cardinality of the power set of S is 8. $|S| = 4$ and ~~so~~ $|\mathcal{P}(S)| = 2^4 = 16$.

A. (ii) and (iv) **B.** only (vi) **C.** (i) and (i)

D. (iv) and (vi) **E.** (i) and (v) **F.** only (iv)

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6. Which of the following arguments (rules of inference) are **invalid**?

(i) $\begin{array}{l} a \to b \\ \neg b \\ \hline \therefore \neg a \end{array}$ (ii) $\begin{array}{l} a \to b \\ \neg a \\ \hline \therefore \neg b \end{array}$ (iii) $\begin{array}{l} a \lor b \\ \neg a \lor c \\ \hline \therefore b \lor c \end{array}$

(iv) $\begin{array}{l} a \lor b \\ \neg b \\ \hline \therefore a \end{array}$ (v) $\begin{array}{l} a \lor b \\ \neg a \lor c \\ \hline \therefore b \land c \end{array}$ (vi) $\begin{array}{l} a \to b \\ \neg a \to c \\ \hline \therefore \neg b \to c \end{array}$

A. only (vi) **B.** (ii) and (v) **C.** (i) and (v) **D.** (iii) and (vi) **E.** (ii) and (iv)

(i) valid (Modus tollens)

(ii) invalid

(iii) valid (Resolution)

(iv) valid (Disjunctive syllogism)

(v) invalid

(vi) valid: $a \to b \equiv \neg b \to \neg a$ (contrapositive)

$\begin{array}{l} \neg b \to \neg a \\ \neg a \to c \\ \hline \therefore \neg b \to c \end{array}$ valid (Hypothetical syllogism)

3 points 7. Let X, Y, and Z be subsets of the universal set U. Use properties of set operations and set identities to show the following. *You need not name the identities used.*

$$X - (\overline{Y} \cap Z) = (X \cap Y) \cup (X - Z)$$

$$X - (\overline{Y} \cap Z) = X \cap \overline{(\overline{Y} \cap Z)}$$ 1P

$$= X \cap (Y \cup \overline{Z})$$ 1P

$$= (X \cap Y) \cup (X \cap \overline{Z})$$ 1P

$$= (X \cap Y) \cup (X - Z)$$

4 points

8. Define the following atomic propositions:

T: "The tiger hides."
S: "The hunt is finished soon."
D: "The tiger is killed."
H: "The hunter is eaten by the tiger."
N: "The hunt is happening at night."

Translate each of the following sentences into compound logical propositions using the atomic propositions T, S, D, H, and N as defined above.

(a) For the hunt to be finished soon, it is necessary that the tiger be killed or the hunter be eaten by the tiger.

1p $S \to D \cup H$

(b) The tiger hides only if the hunt is happening at night.

1p $T \to N$

(c) For the hunt to be finished soon, it is necessary and sufficient that the hunt be happening at night and the hunter be eaten by the tiger.

1p $S \leftrightarrow N \wedge H$

(d) If the tiger hides or the hunt is not happening at night, then (the hunt is not finished soon unless the hunter is eaten by the tiger).

1p $T \vee \neg N \to (\neg H \to \neg S)$

5 points

9. Use any method you know to determine whether or not the argument below is valid. If the argument is not valid, give a counterexample.

H_1 $\neg(C \wedge \neg B)$

H_2 $B \leftrightarrow \neg(C \to B)$

H_3 $A \leftrightarrow C$

C $\therefore \neg(B \to A)$

Construct a truth tree for

$H_1 \wedge H_2 \wedge H_3 \wedge \neg C$.

$\neg(C \wedge \neg B) \equiv \neg C \vee B$ ✓

$B \leftrightarrow \neg(C \to B) \equiv (B \wedge \neg(C \to B)) \vee (\neg B \wedge (C \to B))$

$\equiv (B \wedge \neg(\neg C \vee B)) \vee (\neg B \wedge (\neg C \vee B))$

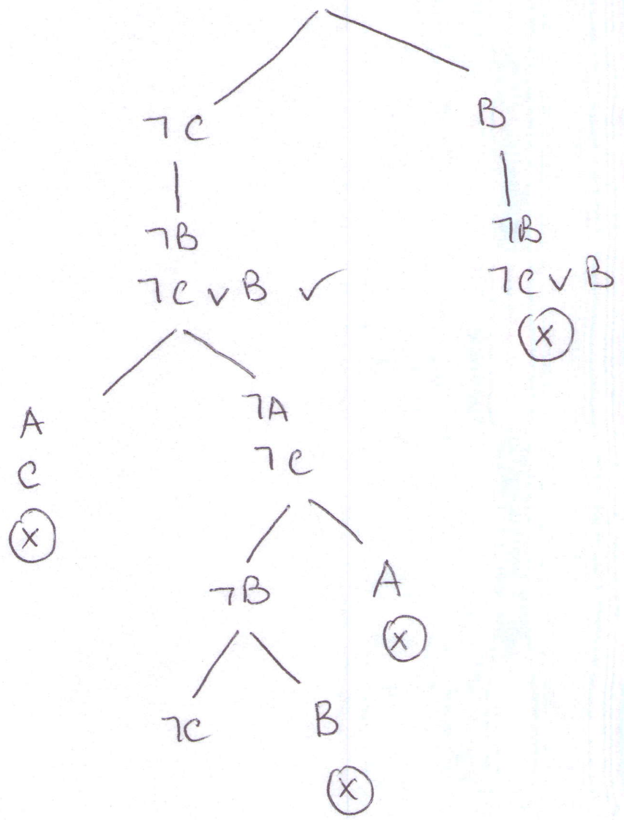
$\equiv (\underline{B} \wedge C \wedge \underline{\neg B}) \vee (\neg B \wedge (\neg C \vee B))$

$\equiv F \vee \underbrace{(\neg B \wedge (\neg C \vee B))}$

$\equiv \neg B \wedge (\neg C \vee B)$ ✓

✓ $(A \wedge C) \vee (\neg A \wedge \neg C) \equiv A \leftrightarrow C$

✓ $\neg B \vee A \equiv \neg(\neg(B \to A))$



4p (1p for the root of the tree)

Since there exists a complete active path, the argument is not valid.

1p Counterexample: A is F, B is F and C is F.

(you can verify that H_1, H_2 and H_3 are all T, but C is F)

4points 10. Let n be an integer. Give an **indirect proof** of the following theorem.

If $n^2 + 2n + 3$ is an odd integer, then n is an even integer.

Propositional variables:

p: "$n^2 + 2n + 3$ is odd"

q: "n is even"

1p Theorem: $p \to q$.

Indirect proof: Assume $\neg q$ is true and prove $\neg p$ is true.

1p

Assume ~~q~~ n is odd. There exists $k \in \mathbb{Z}$ such that $n = 2k + 1$.

Then,

$$n^2 + 2n + 3 = (2k+1)^2 + 2(2k+1) + 3$$
$$= 4k^2 + 4k + 1 + 4k + 2 + 3$$
$$= 4k^2 + 8k + 6$$
$$= 2(2k^2 + 4k + 3)$$

1p

Denote $m = 2k^2 + 4k + 3 \in \mathbb{Z}$ since $k \in \mathbb{Z}$.

1p So, $n^2 + 2n + 3 = 2m$ and hence, $\neg p$ is true.

(Only the mathematical solution is required so do not deduct points for not using propositional variables or ~~and~~ writing the proof strategy)

5 points

11. Is the function $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by

$$F(x, y) = (-2y, x + y)$$

(a) one-to-one?
(b) onto?
(c) a bijection?

Fully justify your answer.

(a) F is one to one if, for $(x_1, y_1) \in \mathbb{Z}\times\mathbb{Z}$ and $(x_2, y_2) \in \mathbb{Z}\times\mathbb{Z}$.

1p

$$F(x_1, y_1) = F(x_2, y_2) \longrightarrow (x_1, y_1) = (x_2, y_2) \text{ is true}$$

$F(x_1, y_1) = F(x_2, y_2)$ means $(-2y_1, x_1 + y_1) = (-2y_2, x_2 + y_2)$

So, $-2y_1 = -2y_2 \Rightarrow y_1 = y_2$
$x_1 + y_2 = x_2 + y_2$ $\Big\rangle$ $x_1 = x_2$.

1p Therefore $(x_1, y_1) = (x_2, y_2)$

F is one-to-one

(b) F is onto if, for every $(a, b) \in \mathbb{Z}\times\mathbb{Z}$, the equation

1p $F(x, y) = (a, b)$ has at least one solution $(x, y) \in \mathbb{Z}\times\mathbb{Z}$.

Let $(a, b) \in \mathbb{Z}\times\mathbb{Z}$ arbitrarily fixed. Then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.

Consider: $F(x, y) = (a, b)$. Then $(-2y, x+y) = (a, b)$ and so

$$\begin{cases} -2y = a & \rightsquigarrow y = -\frac{a}{2} \\ x + y = b & \rightsquigarrow x = b + \frac{a}{2} \end{cases}$$

Take $a = 1$ and $b = 1$. Then $(a, b) = (1, 1) \in \mathbb{Z}\times\mathbb{Z}$. However,

1p the equation $F(x, y) = (1, 1)$ has <u>no</u> solutions in $\mathbb{Z}\times\mathbb{Z}$. ($x = 1 + \frac{1}{2} \notin \mathbb{Z}$ and $y = -\frac{1}{2} \notin \mathbb{Z}$).

So, F is <u>not</u> onto.

(c) F is <u>not</u> a bijection since F is not onto.

1p

Additional work space. Do not detach this page.