Problems: 1) whether a graph can be drawn in the plane so that no 2 edges cross
2) whether there is a one-to-one correspondence between the vertices of 2 graphs that produces a one-to-one correspondence between the edges of the graphs.

2 Graph Terminology

2.1 Adjacency, Incidence, and Degree

Let $G = (V, E)$ be an undirected graph.

Definition 2.1 Vertices $u, v \in V$ are called adjacent or neighbors (in G)

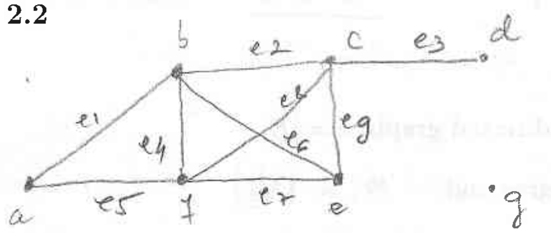
if u and v are endpoints of an edge e of G

We write $e = \{u, v\}$ to say that e connects u and v

An edge $e = \{u, v\}$ is said to be incident with its vertices

u and v.

Example 2.2



$e_1 = \{a, b\}$
$e_2 = \{b, c\}$
$e_3 = \{c, d\}$
$e_4 = \{b, f\}$
$e_5 = \{a, f\}$
$e_6 = \{b, e\}$
$e_7 = \{f, e\}$
$e_8 = \{f, c\}$
$e_9 = \{c, e\}$

Definition: The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$ is called the neighbourhood of v.
If A is a subset of V, denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in
So, $N(A) = \bigcup_{v \in A} N(v)$

Definition 2.3 The degree of a vertex $u \in V$, denoted by $\deg(v)$,

is the number of edges incident with it.

Note that a loop contributes twice to the degree of a vertex it is incident with.

A vertex of degree 0 is called isolated. (an isolated vertex is not adjacent to any vertex)

A vertex of degree 1 is called pendant. (a pendant vertex is adjacent to exactly one vertex).

$\deg(a) = 2$
$\deg(b) = 4$
$\deg(c) = 4$
$\deg(d) = 1$ (d is pendant)
$\deg(e) = 3$
$\deg(f) = 4$
$\deg(g) = 0$ (g is isolated)

sum = 18
$2 \cdot 9 = 18$

(analogy between an edge having 2 endpoints and a handshake involving 2 hands)

Theorem 2.4 [**The Handshaking Theorem**]

undirected

If $G = (V, E)$ is a graph (possibly multigraph or pseudograph) with e edges, then

$$2 \cdot e = \sum_{v \in V} \deg(v)$$

(each edge contributes two to the sum of the degrees of the vertices because it is incident with exactly 2 vertices, possibly equal)

Example 2.5 How many edges in a graph with 5 vertices and degrees 4,3,3,2,2?

Sum of degrees of the verticuses is: $4+3+3+2+2=14$

By Theorem 2.4, $2e = 14$, where e is the number of edges.

Hence, $e = 7$.

Corollary 2.6 An undirected graph has an <u>even</u> number of vertices of <u>odd</u> degree.

PROOF. Let $G = (V, E)$ be an undirected graph, $e = |E|$,

V_1 its set of vertices of odd degree and $v_1 = |V_1|$

V_2 the set of vertices of even degree.

Then, by the Handshaking Theorem,

$$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \underbrace{\sum_{v \in V_2} \deg(v)}$$

even since V_2 contains vertices of even degree

Hence, $\sum_{v \in V_1} \deg(v)$ is even.

But all terms in the sum $\sum_{v \in V_1} \deg(v)$ are odd. $\Rightarrow$

there must be an even number of such terms

(or v_1 is even).

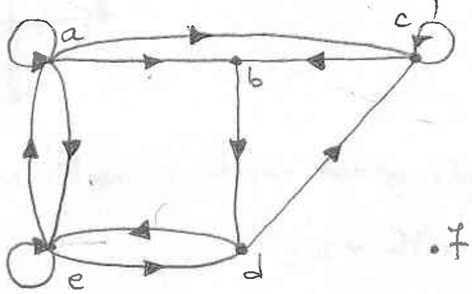
□

2.1.1 Directed Graphs

Let $G = (V, E)$ be a directed graph.

Definition 2.7 If $(u, v) \in E$, then u is said to be adjacent to v
and v is said to be adjacent from u.
u is called the initial vertex of (u, v)
and v is called the terminal or end vertex of (u, v)

Example 2.8 Directed Graph G



$\deg^-(a) = 2$, $\deg^+(a) = 4$
$\deg^-(b) = 2$, $\deg^+(b) = 1$
$\deg^-(c) = 3$, $\deg^+(c) = 2$
$\deg^-(d) = 2$, $\deg^+(d) = 2$
$\deg^-(e) = 3$, $\deg^+(e) = 3$
$\deg(f) = 0$.

Definition 2.9 The in-degree of a vertex u, denoted by $\deg^-(u)$ is the number of edges with u as their terminal vertex.

The out-degree of a vertex u, denoted by $\deg^+(u)$ is the number of edges with u as their initial vertex.

Theorem 2.10 Let $G = (V, E)$ be a directed graph with e edges. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

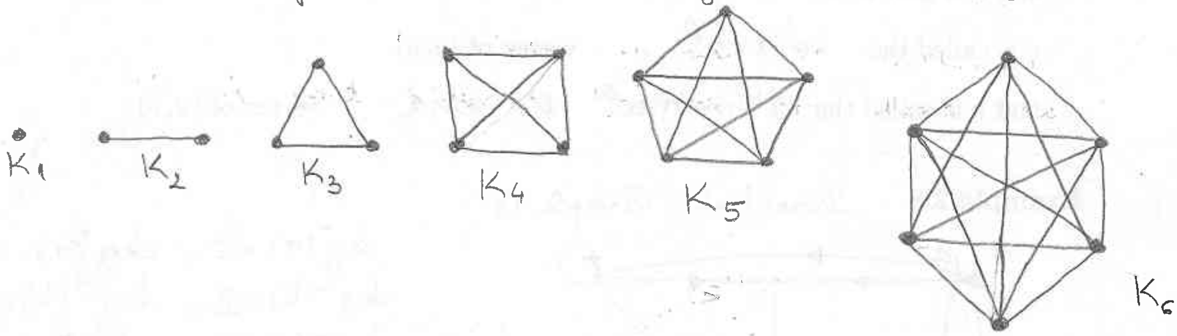
(this is because each edge has an initial and a terminal vertex)

2.2 Some Special Simple Graphs

1. Complete graph with n vertices: K_n - a simple graph that contains exactly one edge between each pair of distinct vertices

   $V = \{ v_1, v_2, \ldots, v_n \}$

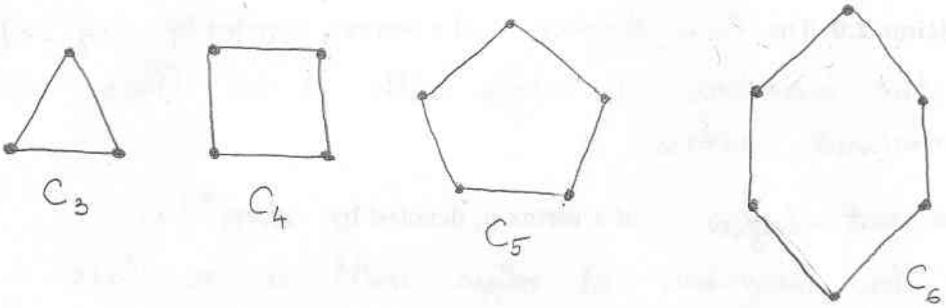
   $E = \{ \{v_i, v_j\}$ such that $v_i \neq v_j$ and $v_i \in V,\ v_j \in V \}$



2. Cycle of length n: C_n - a simple graph with n vertices and n edges

   $V = \{ v_1, v_2, \ldots, v_n \}$, with $n \geq 3$

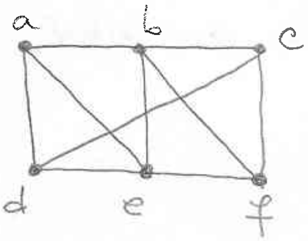
   $E = \{ \{v_i, v_{i+1}\}$, where $v_i \in V$ and $3 \leq i \leq n-1 \}$



3. Path of length n:

   $V = \{ v_1, v_2, \ldots, v_{n+1} \}$

   $E = \{ \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_n, v_{n+1}\} \}$



a, b, c, d, e is a simple path of length 4

d, e, c, a is not a path because $\{e, c\}$ and $\{c, a\}$ are not edges

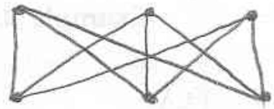
4. Complete bipartite graph with bipartition sets of sizes m and n: $K_{m,n}$

$V = \{ v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n \}$

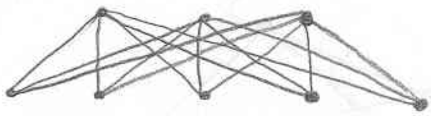
$E = \{ \{ v_i, u_j \}$ such that $v_i \in V$ and $u_j \in V \}$

Note that $K_{m,n}$ has its vertex set partitioned into two sets of vertices V_1 and V_2, with $|V_1| = m$ and $|V_2| = n$ and each edge connects a vertex from V_1 with a vertex from V_2

$K_{2,3}$



$K_{3,3}$

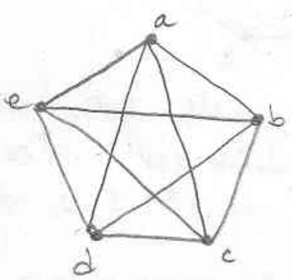


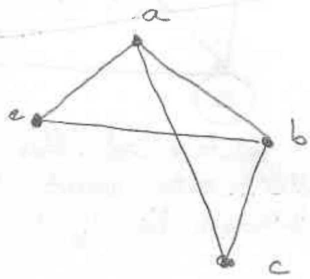
$K_{3,5}$

2.3 Subgraphs

Example 2.11

K_5





Definition 2.12 Let $G = (V, E)$ and $G' = (V', E')$ be graphs.

Then G' is called a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

If $G' \neq G$ we say that G' is a proper subgraph of G.

Example 2.13 In the above example,

$V = \{a, b, c, d, e\}$

$E = \{\{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{c,e\}, \{d,e\}\}$

$V' = \{a, b, c, e\}$

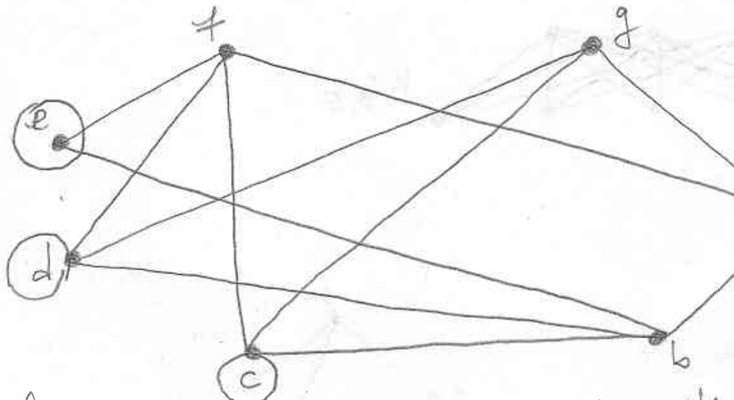
$E' = \{\{a,b\}, \{b,c\}, \{a,c\}, \{b,e\}, \{a,e\}\}$.

2.4 Bipartite Graphs

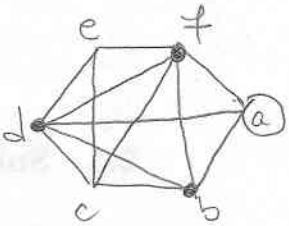
Definition 2.14 A simple graph $G = (V, E)$ is called bipartite if V can be partitioned into two sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and every edge in G connects a vertex in V_1 and a vertex in V_2. (or there is no edge between vertices of subset V_1 and between vertices of subset V_2)

Example 2.15 Is this graph bipartite?

(a)



(b)



We label one vertex of the graph with blue colour: a
The adjacent vertices must have different colour ~~so~~ red: b, f, g
The adjacent vertices to b, f, and g must have other colour: c, d, e
(blue)

Theorem 2.16 The following statements are equivalent for a simple graph $G = (V, E)$:

1. G is bipartite
2. The vertices of G can be coloured with 2 colours so that no two adjacent vertices are assigned the same colour
3. G has no subgraph that is a cycle of odd length

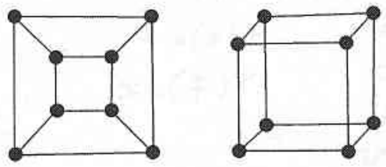
Note that: Example 2.15.

(a) ~~At~~ At the end of the process if all vertices have one unique label, then the graph is bipartite.

(b) d and f have a red colour! a, d, f is a cycle of length 3.

3 Graph Isomorphism. - Section 10.3

Example 3.1



Definition 3.2 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs.

An isomorphism from G_1 to G_2 is a function $\varphi : V_1 \to V_2$ such that φ is bijective and
(u and v adjacent in G_1) $\leftrightarrow$ ($\varphi(u)$ and $\varphi(v)$ adjacent in G_2).

Graphs G_1 and G_2 are called isomorphic if there exists an isomorphism from G_1 to G_2. (there is a bijection between vertices of the two graphs that preserves the adjancy relation)

Note: Isomorphic graphs are "essentially the same", that is, one is obtained from the other by relabelling the vertices. Many graph properties are invariant (i.e. preserved) under isomorphism, e.g.:

- isomorphic simple graphs have the same number of vertices
- isomorphic simple graphs have the same number of edges
- isomorphic simple graphs have equal degrees of vertices

To prove that G_1 and G_2 are isomorphic, we must show that there exist an isomorphism from G_1 to G_2

To prove that G_1 and G_2 are not isomorphic, it suffices to show that they
- either have different numbers of vertices
- either have different numbers of edges
- either have different numbers of vertices of a certain degree