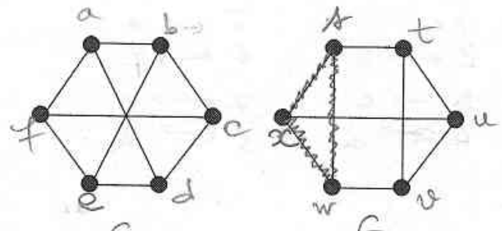
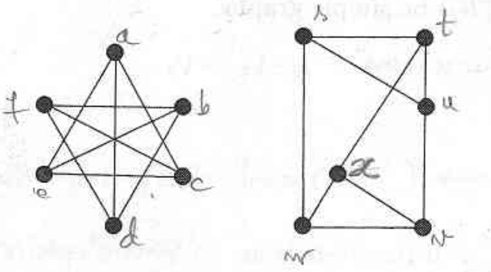
Example 3.3 Are the following pairs of graphs isomorphic?



G_1 G_2

No! Note that G_1 and G_2 have the same number of vertices, edges and vertices' degrees. However G_2 contains a subgraph that is a length 3 cycle, whereas G_1 does not.



H_1 H_2

Yes! let $\varphi: \{a,b,c,d,e,f\} \to \{s,t,u,v,w,x\}$

$\varphi(a)=t$

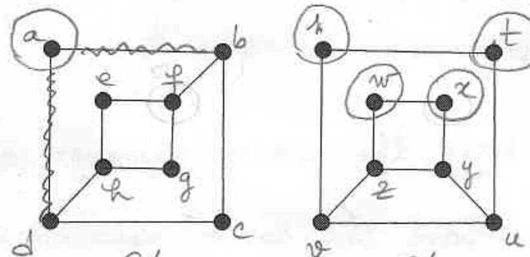
$\varphi(b)=v$

$\varphi(c)=s$

$\varphi(d)=x$

$\varphi(e)=u$

$\varphi(f)=w$

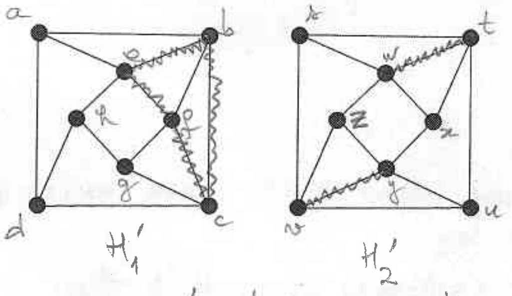


G_1' G_2'

G_1' and G_2' are <u>not</u> isomorphic

- both have 8 vertices
- both have 10 edges
- both have 4 vertices of degree 2 and 4 vertices of degree 3

However, a is adjacent to 2 vertices of degree 3 and deg a = 2. So, a must be mapped into either s, t, x or w. <u>But</u> each of these vertices is adjacent to a vertex of degree 2.



H_1' H_2'

No!

In H_1' there are 4 edges connecting 4 degree vertices: f, e, b, c

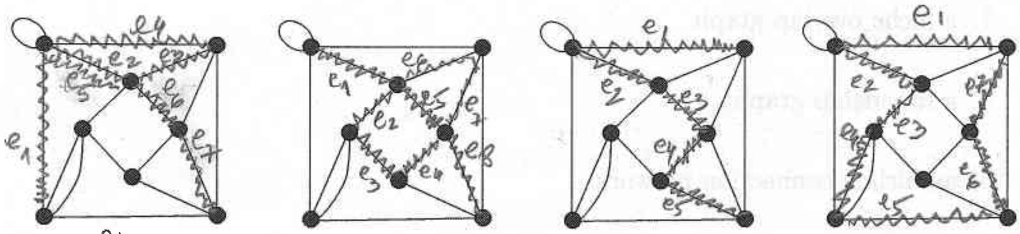
In H_2' there are only 2 edges connecting 4 degree vertices: y, v, w, t

4 Walks, Trails, Paths, Cycles. Connectivity.

Note that the terminology of this section differs from the textbook. A conversion glossary is provided below.

4.1 Walks, Trails, Paths, Cycles

4.1.1 Undirected graphs



walk $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ trail $e_1, e_2, e_3, \dots, e_8$ path $e_1 e_2 e_3 e_4 e_5$ cycle $e_1, e_2, e_3, e_4, e_5, e_6, e_7$

Definition 4.1 Let $G = (V, E)$ be an undirected graph (possibly multigraph or pseudograph) with the function $f : E \to \{\{u, v\} : u, v \in V\}$. Let $n \in \mathbb{N}$.

A walk of length n is a sequence of edges $e_1, e_2, \dots, e_n$ of G such that $f(e_i) = \{x_{i-1}, x_i\}$, $x_i \in V$ and $i = \overline{1,n}$

A walk is called a trail if its edges are pairwise distinct.

A walk is called a path if its (internal) vertices (and therefore edges) are pairwise distinct. (i.e. the path is also a trail)

A walk with $x_0 = x_n$ is called closed walk.

A trail with $x_0 = x_n$ is called closed trail.

A path with $x_0 = x_n$ is called cycle.

Terminology conversion table

Our term	*Rosen's term*	**Our term**	*Rosen's term*
walk	*path*	**closed walk**	*circuit*
trail	*simple path*	**closed trail**	*simple circuit*
path	none	**cycle**	none

Note: In a simple graph, a walk $e_1, e_2, \ldots, e_n$ with $e_1 = \{x_0, x_1\}, e_2 = \{x_1, x_2\}, \ldots, e_n = \{x_{n-1}, x_n\}$ can be more simply denoted by its sequence of vertices:

$x_0, x_1, \ldots, x_{n-1}, x_n$

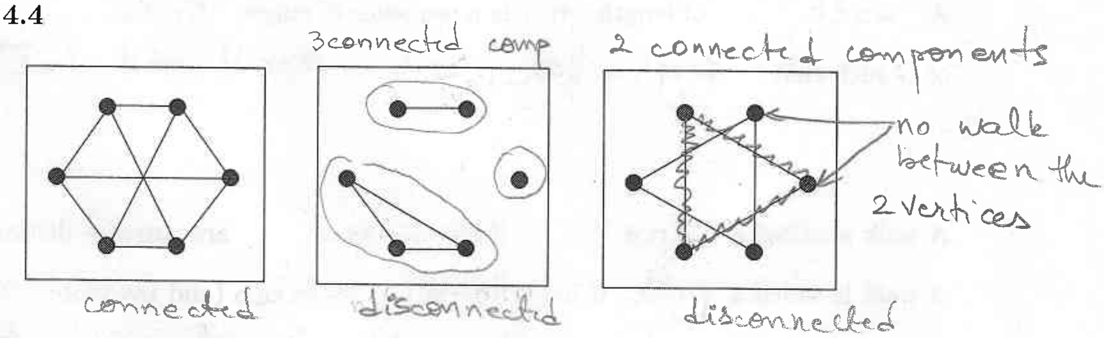
Example 4.2 What can you say about vertices u and v in the following graphs if you know that there exists a path from u to v?

1. a niche overlap graph
2. a friendship graph
3. an airline connection network

4.2 Connectivity in undirected graphs

Definition 4.3 A graph G is called connected if there exists a walk between every pair of vertices in G.

Example 4.4



Definition 4.5 Maximal connected subgraphs of G are called connected components.

Note: Every graph is a collection of its connected components.

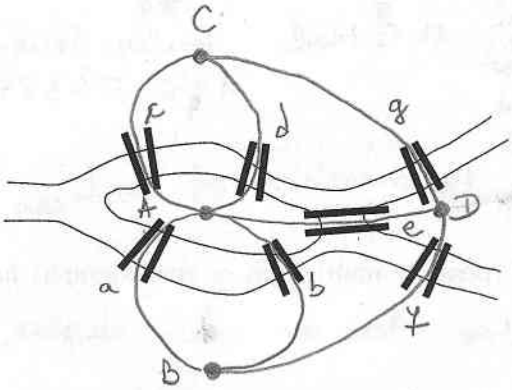
Number of connected components in a connected graph: 1 (see first figure)

Theorem 4.6 Let $G = (V, E)$ be a graph and $u, v \in V$. In G, there exists a ___ a walk from u to v if and only if there exists a ___ a path from u to v

5 Euler trails and tours

Example 5.1 The origins of graph theory — the Bridges of Königsberg

In 1736, while stationed in St. Petersburg, the Swiss mathematician Leonhard Euler took interest in the following puzzle. The townsfolk of Königsberg, Prussia (now Kaliningrad, Russia) take long Sunday walks. They wonder if it is possible to walk around the town, traverse each of the seven bridges of Königsberg exactly once and return to the starting point.



Graph model:

$V = \{A, B, C, D\}$

$E = \{a, b, c, d, e, f, g\}$

vertex: each land mass

edge: connecting vertices where there is a bridge

This is equivalent to the following question:

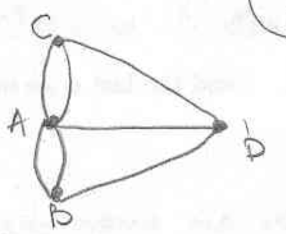
Does there exist a closed trail covering every edge?

Definition 5.2 An Euler tour in a graph G is a closed trail in G containing every edge of G.

An (open) Euler trail in a graph G is a trail (i.e. no edges are repeated) in G containing every edge of G.

So: does the graph K above contain an Euler tour / Euler trail?

No Euler tour.

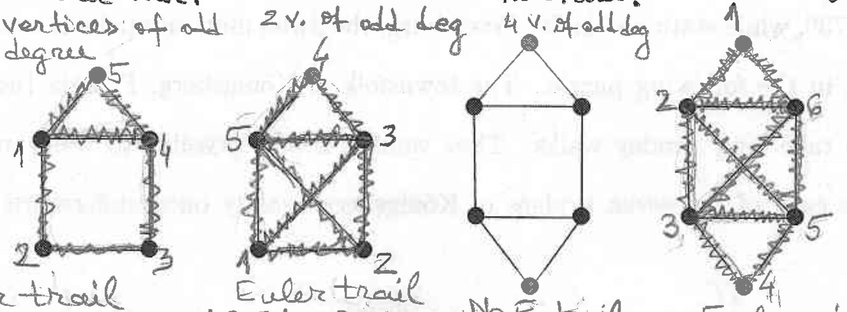


Idea: Whenever we follow a trail in a graph and pass through a vertex, we use two edges at that vertex.

So if we want a trail that uses every edge, then every vertex must have an even degree apart perhaps from the start and finish vertices.

Open Euler trail? NO! (4 vertices of odd degree, i.e more than 2)

Example 5.3 For each the following graphs, determine if it has an Euler tour. If not, does it have an (open) Euler trail?



Conclusion: Start by counting the number of vertices of odd degree.

Theorem 5.4 A connected graph G (possibly multigraph or pseudograph) has an Euler tour if and only if every vertex has an even degree.

PROOF. Suppose a graph $G = (V, E)$ has an Euler tour

$$v_0 v_1 v_2 \dots v_{n-1} v_n v_0$$

First we note that G must be connected, since the tour contains a path between any pair of vertices of G. (i.e. for any $0 \le i \le j \le n$ there exist $v_i v_{i+1} \dots v_j$)

Each time a vertex v_i, $0 \le i \le n$ is visited by the tour, 2 edges of the graph are used up, contributing 2 to the degree of v_i. Hence $\deg(v_i)$ is even for all internal vertices of the tour.

For v_0, the initial edge of the tour adds 1 to $\deg(v_0)$, each visit of v_0 as an internal vertex of the tour adds 2, and the last edge of the tour adds 1.

Hence $\deg(v_0)$ is also even

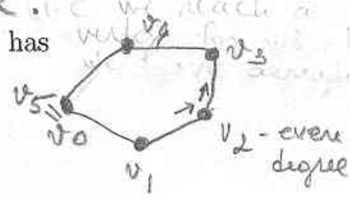
Conclusion: Every vertex has an even degree.

Conversely, suppose G is connected and that every vertex has an even degree.

We construct an Euler tour in G as follows:

1. Let $T: v_0 v_1 v_2 \dots v_{n-1} v_n$ be a trail of G. T can be constructed as follows: Choose an arbitrary vertex of G, v_0 and an edge incident with v_0 (this can be done since G is connected), v_1. Continue to add edges to $\{v_0, v_1\}$ one by one $\{v_1, v_2\} \dots \{v_{n-1}, v_n\}$ until you cannot add another edge to the trail.

   Since T is a (maximal) trail in G and every vertex of G has even degree, T must in fact be a closed trail.



2. If T uses all edges of G, then T is the required Euler tour.

3. Otherwise, remove the edges of T from G to obtain a graph G'. (Note that G' has only even degree vertices)

   Let T' be a trail of G'.

   Then T' must in fact be a closed trail.

4. Since G is connected, T and T' must have a common vertex v.

5. Join T and T' into T''.

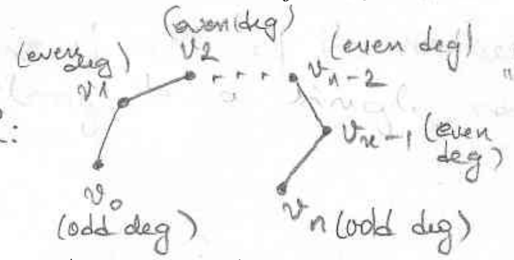
   Replace T by T''.

6. Repeat Steps 2–5.

Note that since G has a finite number of vertices, this algorithm indeed terminates, producing an Euler tour. □

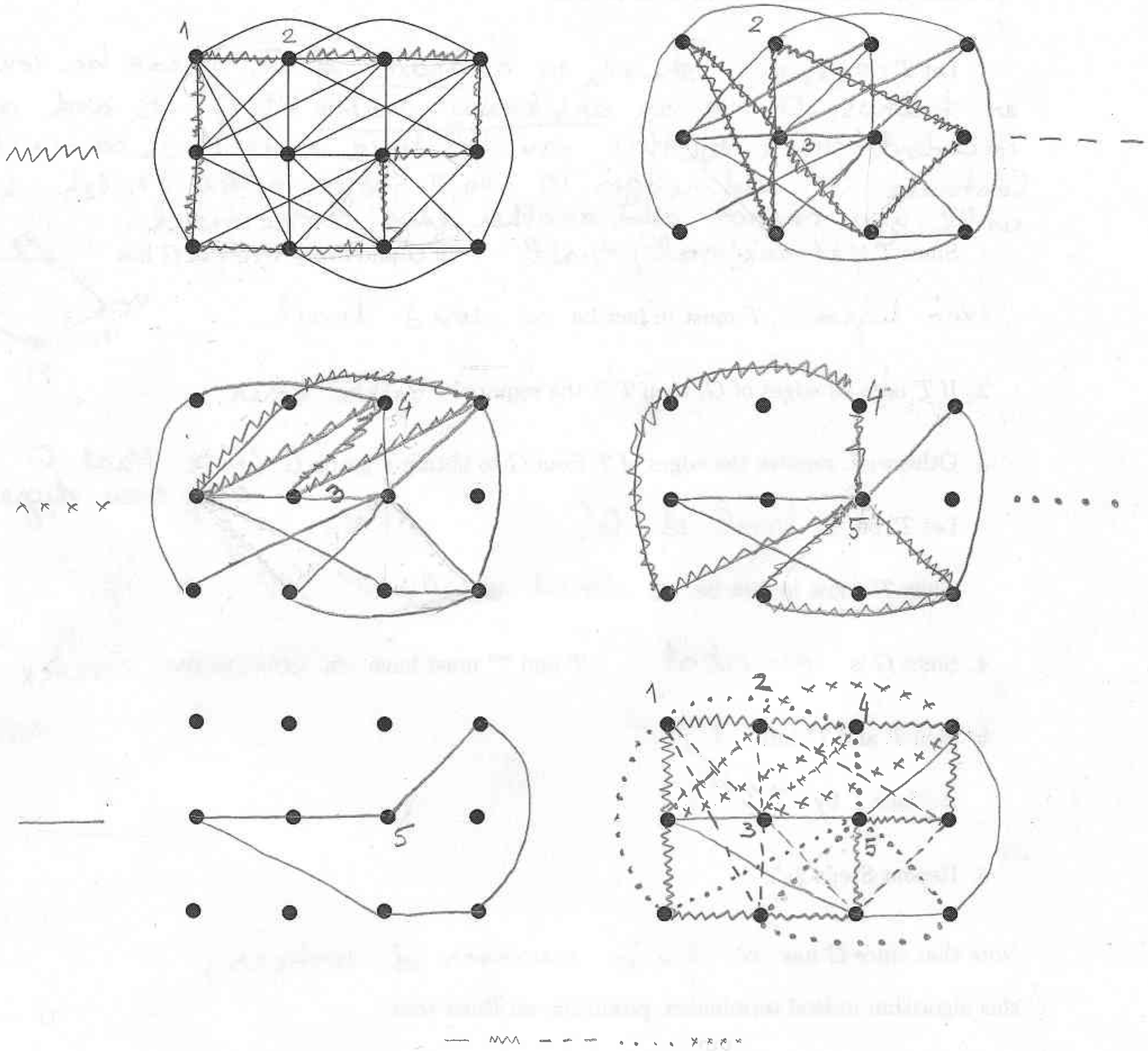
Corollary 5.5 A connected graph G (possibly multigraph or pseudograph) has an open Euler trail if and only if it has exactly 2 vertices of odd degree.

(open) Euler trail: $v_0 \neq v_n$



$\Leftarrow$ " Assume it has exactly 2 vertices of odd degree v_0 and v_n. Connect them and obtain a larger graph, where every vertex has even degree. Then there exist an Euler tour (by Theorem) This implies that the removal of edge $\{v_0, v_n\}$ will generate an open Euler trail in the original graph.

Example 5.6 Use the algorithm from the proof of Theorem 5.4 to construct an Euler tour of the graph below.



1 2 3 4 5 were used as connecting vertices