Problem 1.

$\{w_1, w_2, w_3\}$ $\{b_1, b_2\}$

Two balls are selected at random from a bag with three white balls and two black balls. For each of the following enter the probability in the accompanying box.

- The probability the first ball is white. $\boxed{\frac{12}{20} = \frac{3}{5}} = \frac{\left(\frac{(3C1)(4C1)}{2!}\right)}{(5C2)}$
- The probability the second ball is white. $\boxed{\frac{12}{20} = \frac{3}{5}} =$ || (symmetry)
- The probability both balls are white. $\boxed{\frac{6}{20} = \frac{3}{10}} = \frac{(3C2)}{(5C2)}$
- The probability the second ball is white, given that the first ball is white. $\boxed{\frac{6}{10} = \frac{1}{2}} = \frac{(2C1)}{(4C1)}$

$$"S" = \left\{\begin{matrix} (b_1,b_2), & (b_1,w_1), & (b_2,w_1), & (w_1,b_1), & (w_1,w_2) \\ (b_2,b_1), & (b_1,w_2), & (b_2,w_2), & (w_1,b_2), & (w_1,w_3) \\ & (b_1,w_3), & (b_2,w_3), & (w_2,b_1), & (w_2,w_1) \\ & & & (w_2,b_2), & (w_2,w_3) \\ & & & (w_3,b_1), & (w_3,w_1) \\ & & & (w_3,b_2), & (w_3,w_2) \end{matrix}\right\}$$

NOTE: The questions lead us to believe that it is a good idea to distinguish (b,w) from (w,b), but it is NOT NECESSARY!

Problem 2.

- An insurance company has these data: The probability of an insurance claim in a period of one year is 4% for persons under age 30 and 2% for persons over age 30. It is also known that 30% of the targeted population is under age 30. What is the probability of an insurance claim in a period of one year for a randomly chosen person from the targeted population? Enter your answer in the box below.

U: under 30 ; $P(U) = 0.30$

I: insurance claim ; $\begin{cases} P(I \mid U) = 0.04 \\ P(I \mid \bar{U}) = 0.02 \end{cases}$

$\boxed{P(I) = 0.026 = 26/1000}$

$$P(I) = P(I \mid U) \cdot P(U) + P(I \mid \bar{U}) \cdot P(\bar{U})$$

- Suppose 1 in 1000 persons has a certain disease. A test detects the disease in 99 % of diseased persons. The test also "detects" the disease in 1 % of healthly persons. With what probability does a positive test diagnose the disease? Enter your answer in the box below.

D: disease; $P(D) = 1/1000$

$+$: test positive; $\begin{cases} P(+ \mid D) = 0.99 \\ P(+ \mid \bar{D}) = 0.01 \end{cases}$

$\boxed{P(D \mid +) = 99/1098}$

$$P(D \mid +) = \frac{P(+ \mid D) \cdot P(D)}{(P(+ \mid D) \cdot P(D) + P(+ \mid \bar{D}) \cdot P(\bar{D}))}$$

Problem 3. Given the joint probability mass function in the Table below on the left:

Probability mass function $p_{X,Y}(x,y)$

	$Y=0$	$Y=1$	$Y=2$	$p_X(\cdot)$
$X=0$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	1/2
$X=1$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	1/3
$X=2$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	1/6
$p_Y(\cdot)$	1/4	1/4	1/2	1

Probability distribution function $F_{X,Y}(x,y)$

	$Y\leq 0$	$Y\leq 1$	$Y\leq 2$	$F_X(\cdot)$
$X\leq 0$	1/8	1/4	1/2	1/2
$X\leq 1$	5/24	5/12	5/6	5/6
$X\leq 2$	1/4	1/2	1/1	1/1
$F_Y(\cdot)$	1/4	1/2	1/1	

- Fill in the marginal probability mass function values.
- Fill in the probability distribution Table.

Problem 4. Given the Tables in Problem 3, determine the following:

- Are X and Y independent?

$\boxed{\text{YES}}$

X & Y are ind't

iff $p_{(X,Y)}(x,y) = p_X(x) \cdot p_Y(y) \quad \forall x, y$;

You can quickly see that

$p(x,y) = p(x) \cdot p(y) \quad \forall x, y$.

$\Rightarrow X$ & Y are ind't!

- $P(X \leq 1) = F_X(1)$

$\boxed{5/6}$

- $P(X \leq 1\,,\, 0 < Y \leq 2)$

$\boxed{5/8}$

$= P(X \leq 1) \cdot P(0 < Y \leq 2)$ [Ind.]

$= F_X(1) \cdot [P(0 \leq Y \leq 2) - P(Y=0)]$

$= F_X(1)(F_Y(2) - P(Y=0))$

$= \frac{5}{6}\left(1 - \frac{1}{4}\right) = \frac{5}{8}$ (OR, just look at the 2nd table !!!)

- $P(X = 2 \mid Y = 2) = P(X=2)$ [Ind.]

$\boxed{1/6}$

Problem 5. Given the Tables in Problem 3, determine the following:

- $E[X]$ and $E[Y]$

$$\boxed{E[X] = \frac{2}{3}, \quad E[Y] = \frac{5}{4}}$$

$$E[X] = \sum_x x \cdot p(x)$$
$$= 0 \cdot P_X(0) + 1 \cdot P_X(1) + 2 \cdot P_X(2)$$
$$= 0 + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6}$$
$$E[Y] = 0 \cdot P_Y(0) + 1 \cdot P_Y(1) + 2 \cdot P_Y(2)$$
$$= 0 + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2}$$

- $Var[X] = E[X^2] - (E[X])^2$

$$\boxed{Var(X) = \frac{5}{9}}$$

$$= \sum_x x^2 \cdot p(x) - \left(\frac{2}{3}\right)^2$$
$$= \left(0 + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{6}\right) - \frac{4}{9}$$
$$= \frac{1}{3} + \frac{4}{6} - \frac{4}{9} = \frac{5}{9}$$

- $E[XY] = E[X] \cdot E[Y]$ (Ind.)

$$\boxed{E[XY] = 5/6}$$

- $cov(X, Y) = 0$ (Ind)

$$\boxed{cov(X,Y) = 0}$$

$$= E[XY] - E[X] \cdot E[Y]$$

Problem 6. A die is rolled five times. A roll is considered a "success" if the die lands with a six up. An example of an outcome is then 10010 , where "1" denotes "success" and "0" "failure".

Binomial $(n=5, p=1/6)$

$X = \#$ successes, $X \in \text{Bin}(n=5, p=\frac{1}{6})$

- What is the probability of the outcome 10010 ?

$$P(10010) = \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}$$
$$= 5^3/6^5$$

$\boxed{125/7776}$

- What is the probability of exactly two successes?

$$P(X=2) = \binom{5}{2}\cdot p^2\cdot(1-p)^3$$
$$= 10\cdot\left(\frac{1}{6}\right)^2\cdot\left(\frac{5}{6}\right)^3 = 10\cdot\frac{125}{7776}$$

$\boxed{P(X=2) = \frac{625}{3888}}$

- What is the probability of one or more successes?

$$P(X\geq 1) = 1 - P(X<1)$$
$$= 1 - P(X=0)$$
$$= 1 - \binom{5}{0}\cdot p^0\cdot(1-p)^5$$
$$= 1 - \left(\frac{5}{6}\right)^5$$

$\boxed{P(X\geq 1) = \frac{4651}{7776}}$

- What is the probability of less than five successes?

$$P(X<5) = 1 - P(X\geq 5)$$
$$= 1 - P(X=5)$$
$$= 1 - \binom{5}{5}\cdot p^5\cdot(1-p)^0 = 1 - \frac{1}{6^5}$$

$\boxed{P(X<5) = \frac{7775}{7776}}$

Problem 7. In each of the 4 parts of this problem your answer may be a *numerical expression* , rather than the final numerical probability:

Suppose that customers arrive at a counter at the rate of 10 per hour.

Assuming that the arrivals have a Poisson distribution, what is the probability that :

X = # arrivals/ hour
$X \in \text{Poiss}(\lambda = 10)$

- no customer arrives in an hour?

$$P(X=0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-10}$$

$\boxed{P(X=0) \cong 0.0000454}$

- ten customers arrive in an hour?

$$P(X=10) = e^{-\lambda} \cdot \frac{\lambda^{10}}{10!} = e^{-10} \cdot \frac{10^{10}}{10!}$$

$\boxed{P(X=10) \cong 0.125}$

Also give the above probabilities using the binomial random variable, after by dividing the hour into 60 intervals of one minute :

$X \in \text{Bin}\left(n = 60,\ p = \frac{10}{60}\right);\ \lambda = np = 10$

- no customer arrives in an hour?

$$P(X=0) = \binom{60}{0} \cdot p^0 \cdot (1-p)^{60}$$

$\boxed{P(X=0) \cong 0.0000177}$

- ten customers arrive in an hour?

$$P(X=10) = \binom{60}{10} \cdot p^{10} \cdot (1-p)^{50}$$

$\boxed{P(X=10) \cong 0.137}$

Problem 8. For the random variable X with probability density function

$$f(x) = \begin{cases} c\,x^2(1-x) & , \quad 0 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

- Show that c must have the value $c = 12$.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 c \cdot x^2 (1-x)\,dx = c \cdot \int_0^1 (x^2 - x^3)\,dx$$

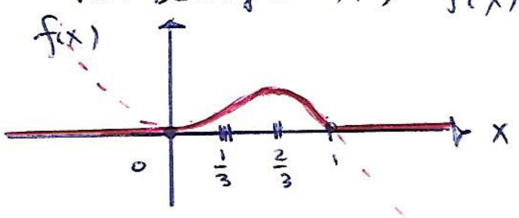
$$= c \cdot \left(\frac{x^3}{3} - \frac{x^4}{4} \right)\Big|_{x=0}^{x=1} = c \cdot \left(\frac{1}{3} - \frac{1}{4} \right) = c \cdot \frac{1}{12}$$

$$\Rightarrow \quad c = 12$$

- Draw an accurate graph of $f(x)$.

· $f(x) = 12 \cdot x^2 (1-x)$ has zeros at $x = 0$ & $x = 1$.

· For large x, $f(x) \cong -x^3 \Rightarrow \begin{cases} f \to -\infty & \text{as } x \to \infty \\ f \to +\infty & \text{as } x \to -\infty \end{cases}$



· $f'(x) = 0 \Rightarrow 2x - 3x^2 = x(2-3x) = 0$

$\Rightarrow x = 0$ or $x = 2/3$

$\therefore$ loc. max at $x = 2/3$

- What is the value of $E[X]$?

$$\boxed{E[X] = 3/5}$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= 12 \cdot \int_0^1 x^3 (1-x)\,dx$$

$$= 12 \cdot \left(\frac{x^4}{4} - \frac{x^5}{5} \right)\Big|_{x=0}^{x=1} = 12 \cdot \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{12}{20}$$

- Determine the distribution function $F(x)$.

$$\boxed{F(x) = x^3(4-3x) \text{ if } 0 \le x \le 1.}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

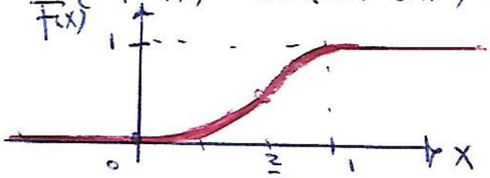
$$= \begin{cases} 0 & x < 0 \\ \int_0^x 12 \cdot t^2 \cdot (1-t)\,dt = 12 \cdot \left(\frac{t^3}{3} - \frac{t^4}{4} \right)\Big|_{t=0}^{t=x} = 12 \cdot \left(\frac{x^3}{3} - \frac{x^4}{4} \right), & 0 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

i.e., $$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3(4-3x) & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \end{cases}$$

Problem 9. For the random variable X in Problem 8 :

- Draw the graph of $F(x) := x^3(4-3x)$ f $0 \le x \le 1$

$0 \le x \le 1$: $\begin{cases} F'(x) = 12x^2 - 12x^3 = 12x^2(1-x) \geq 0 \;\forall\; 0 \le x \le 1 \Rightarrow F' \text{ is always increas.} \\ F''(x) = 12(2x-3x^2) = 12x(2-3x) = 0 \text{ at } x = \frac{2}{3} : \text{pt of inflection} \end{cases}$



- What is $P(X \le \frac{1}{2})$?

$P(X \le \frac{1}{2}) = F(\frac{1}{2}) = (\frac{1}{2})^3 \cdot (4 - 3 \cdot \frac{1}{2})$

$\boxed{P(X \le \frac{1}{2}) = 5/16}$

- Compute $E[X^2]$.

$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 12 \cdot x^4 (1-x)\,dx$

$= 12\left(\frac{x^5}{5} - \frac{x^6}{6}\right)\Big|_{x=0}^{x=1} = 12\left(\frac{1}{5} - \frac{1}{6}\right)$

$\boxed{E[X^2] = 2/5}$

- Compute the standard deviation $\sigma(X)$. $= \sqrt{Var(X)}$

$Var(X) = E[X^2] - E[X]^2$

$= \frac{2}{5} - \left(\frac{3}{5}\right)^2$

$= 1/25 = \sigma^2(X)$

$\boxed{\sigma(X) = 1/5}$

Problem 10.

Use the method of moments to compute the mean and the variance of the exponential random variable X with density function

$$f(x) = e^{-x}, \text{ for } x \geq 0 .$$

Show all details of your work.

1. Find $E[e^{tx}] =: m(t)$
2. Find $m'(t)$ & $m''(t)$
3. Evaluate $m'(t), m''(t)$ at $t = 0$.

1. $$m(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$
$$= \int_0^{\infty} e^{tx} \cdot e^{-x}\, dx = \int_0^{\infty} e^{x(t-1)} dx$$
$$= \frac{1}{t-1} \cdot e^{x(t-1)} \Big|_{x=0}^{x\to+\infty} \quad \text{iff } t<1$$ (otherwise, the integral does not exist!)

i.e., $t<1$ w/ $$m(t) = \frac{1}{t-1}\left(\lim_{x\to\infty} e^{x(t-1)} - e^0\right)$$
$$= \frac{1}{t-1}(0-1)$$ (Since $t<1$, $t-1<0$ $\Rightarrow e^{x(t-1)} \simeq e^{-x}$ for x large)
$$= -1/(t-1) = -(t-1)^{-1}$$

2. $$\left.\begin{aligned} m'(t) &= -1[-1(t-1)^{-2}] = (t-1)^{-2} \\ m''(t) &= -2(t-1)^{-3} \end{aligned}\right\} \quad t<1$$

3. $$m'(0) = (0-1)^{-2} = 1/(-1)^2 = 1 = E[X]$$
$$m''(0) = -2(0-1)^{-3} = -2/(-1)^3 = -2/-1 = 2 = E[X^2]$$

$$\Rightarrow \begin{cases} E[X] = 1 \\ Var(X) = E[X^2] - (E[X])^2 = 2 - 1^2 = 1 \end{cases}$$

∴, if $X \in Exp(\lambda = 1)$, $E[X] = Var(X) = \lambda = 1$.