

Mathematics 263, Winter 2005
Ordinary Differential Equations and Linear Algebra
Solutions to the midterm, Version 1

1. Solve for $y(x)$.

$$y' - 2xy = x, \quad y(0) = 1.$$

Solution: This is first-order linear. An integrating factor is $e^{\int (-2x)dx} = e^{-x^2}$. Multiplying the equation by this gives

$$(e^{-x^2}y)' = e^{-x^2}y' - 2xe^{-x^2}y = xe^{-x^2},$$

so $e^{-x^2}y = \int xe^{-x^2}dx + C = -\frac{1}{2}e^{-x^2} + C$. (We use the substitution $u = -x^2$, $du = -2xdx$ for the integral.) So the general solution is

$$y = -\frac{1}{2} + Ce^{x^2}.$$

As $y(0) = 1$, $1 = -\frac{1}{2} + C$, $C = \frac{3}{2}$ and the particular solution for which we so yearn is

$$y(x) = -\frac{1}{2} + \frac{3}{2}e^{x^2}.$$

OR: The equation is also separable. We can rewrite it as $y' = x(1 + 2y)$ or

$$\frac{y'}{1 + 2y} = x.$$

Integrating both sides gives

$$\frac{1}{2} \ln |1 + 2y| = \frac{1}{2}x^2 + C, \text{ or}$$

$$1 + 2y = Ke^{x^2}, \text{ or}$$

$$y = -\frac{1}{2} + \frac{K}{2}e^{x^2}.$$

Plugging in the initial condition gives us $K = 3$ and the same particular solution as above.

2. Solve for $y(x)$.

$$y' = \frac{x^2 + 3y^2}{2xy}, \quad y(1) = 1.$$

Solution: This is homogeneous. Letting $y = ux$, so $y' = u'x + u$, the equation becomes

$$u'x + u = \frac{x^2 + 3u^2x^2}{2x^2u} = \frac{1 + 3u^2}{2u},$$

which is now separable. We rewrite it as

$$\frac{2uu'}{1+u^2} = \frac{1}{x}$$

and integrate both sides to get $\ln(1+u^2) = \ln|x| + C$ and then $1 + (\frac{y}{x})^2 = Kx$, so the general solution is

$$y(x) = \pm x\sqrt{Kx-1}.$$

As $y(1) = 1$, the sign is positive and $K = 2$. The particular solution is

$$y(x) = x\sqrt{2x-1}.$$

(It's valid and unique for $x > \frac{1}{2}$, incidentally.)

3. Solve implicitly.

$$y' = \frac{4x^2 + 3 \cos y}{x \sin y}, \quad y(1) = \frac{\pi}{2}.$$

Solution: If we rewrite the equation as

$$(4x^2 + 3 \cos y)dx - x \sin y dy = 0.$$

It is not exact, so we look for an integrating factor. With $M = 4x^2 + 3 \cos y$ and $N = -x \sin y$, we have $M_y = -3 \sin y$ and $N_x = -\sin y$.

$$\frac{M_y - N_x}{N} = \frac{-2 \sin y}{-x \sin y} = \frac{2}{x}$$

and an integrating factor is $e^{\int \frac{2dx}{x}} = e^{2 \ln x} = x^2$. Multiplying the equation by this yields

$$(4x^4 + 3x^2 \cos y)dx - x^3 \sin y dy = 0,$$

and this is now exact. Integrating the expression in front of dx in terms of x gives $\frac{4}{5}x^5 + x^3 \cos y$; the y -derivative of this is already $-x^3 \sin y$, so the general solution is

$$\frac{4}{5}x^5 + x^3 \cos y = C.$$

Plugging in our initial condition gives the particular solution

$$\frac{4}{5}x^5 + x^3 \cos y = \frac{4}{5}.$$

4. Solve for $y(x)$.

$$y'' = x[y']^2, \quad y(0) = 1, \quad y'(0) = 0.5.$$

Solution: The substitution $z = y'$ turns the equation into the (first-order, separable) oddity $z' = xz^2$ or

$$\frac{z'}{z^2} = x.$$

Integrating both sides donates $-\frac{1}{z} = \frac{1}{2}x^2 + C$, or

$$z = \frac{-2}{x^2 + 2C}.$$

Since $y'(0) = z(0) = 0.5$ we see that $C = -2$ and we are now solving $y' = \frac{-2}{x^2-4}$. Now

$$\frac{-2}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

when $A(x+2) + B(x-2) = -2$, that is when $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Integrating we get

$$y(x) = -\frac{1}{2} \ln|x-2| + \frac{1}{2} \ln|x+2| + D.$$

Now our initial condition tells us that $D = 1$, so the particular solution our heart pines for is

$$y(x) = -\frac{1}{2} \ln|x-2| + \frac{1}{2} \ln|x+2| + 1.$$

5. Solve for $y(x)$.

$$D^4y + 4D^2y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0.$$

Solution: The characteristic polynomial is $r^4 + 4r^2 = r^2(r^2 + 4)$. Its roots are 0 (of multiplicity two) and $\pm 2i$. So the general solution is

$$y(x) = c_1 + c_2x + c_3 \cos(2x) + c_4 \sin(2x).$$

$y(0) = 1$ says that $c_1 + c_3 = 1$.

$$y'(x) = c_2 - 2c_3 \sin(2x) + 2c_4 \cos(2x),$$

so $y'(0) = 0$ says that $c_2 + 2c_4 = 0$.

$$y''(x) = -4c_3 \cos(2x) - 4c_4 \sin(2x)$$

and therefore $y''(0) = 1$ announces $-4c_3 = 1$.

$$y'''(x) = 8c_3 \sin(2x) - 8c_4 \cos(2x)$$

and our last initial condition tells us that $y'''(0) = 0 = -8c_4$. So $c_4 = 0$, $c_3 = -\frac{1}{4}$, $c_2 = 0$ and $c_1 = \frac{5}{4}$. The celebrated particular solution is

$$y(x) = \frac{5}{4} - \frac{1}{4} \cos(2x).$$

6. Find the general solution $y(x)$ of

$$y'' + 6y' + 9y = \frac{1}{2}e^{-3x} + \sin x.$$

Solution: The characteristic polynomial for the homogeneous equation is $r^2 + 6r + 9 = (r + 3)^2$ with root -3 of multiplicity two. Hence the general solution of the homogeneous equation is

$$y_h = c_1e^{-3x} + c_2xe^{-3x}.$$

We use undetermined coefficients to find a particular solution, but because e^{-3x} and xe^{-3x} solve the homogeneous equation, our solution will have the form

$$y_p = Ax^2e^{-3x} + B \cos x + C \sin x.$$

$$\text{Then } y'_p = e^{-3x}(-3Ax^2 + 2Ax) - B \sin x + C \cos x$$

$$\text{and } y''_p = e^{-3x}(9Ax^2 - 12Ax + 2A) - B \cos x - C \sin x.$$

So

$$\begin{aligned} y''_p + 6y'_p + 9y_p &= e^{-3x}(9Ax^2 - 12Ax + 2A - 18Ax^2 + 12Ax + 9Ax^2) + (8B + 6C) \cos x + (-6B + 8C) \sin x \\ &= 2Ae^{-3x} + (8B + 6C) \cos x + (-6B + 8C) \sin x. \end{aligned}$$

So $A = \frac{1}{4}$, $8B + 6C = 0$ and $-6B + 8C = 1$; solving gives $B = -\frac{3}{50}$, $C = \frac{2}{25}$.

$$y_p = \frac{1}{4}x^2e^{-3x} - \frac{3}{50} \cos x + \frac{2}{25} \sin x$$

and the general solution is

$$y(x) = c_1e^{-3x} + c_2xe^{-3x} + \frac{1}{4}x^2e^{-3x} - \frac{3}{50} \cos x + \frac{2}{25} \sin x.$$

7. Find the general solution $y(t)$ of

$$y'' + 3y' + 2y = \frac{1}{1 + e^t}.$$

Solution: The characteristic polynomial of the homogeneous equation is $r^2 + 3r + 2$ with roots $-1, -2$ and so the general solution to the homogeneous equation is

$$y_h = c_1e^{-t} + c_2e^{-2t}.$$

We employ variation of parameters with $y_1 = e^{-t}$ and $y_2 = e^{-2t}$ to find $y_p = u_1y_1 + u_2y_2$. The Wronskian $W(y_1, y_2)$ is $e^{-t}(-2e^{-2t}) - e^{-2t}(-e^{-t}) = -e^{-3t}$. So

$$u'_1 = -\frac{y_2g(t)}{W} = \frac{-e^{-2t}}{-e^{-3t}} \frac{1}{1 + e^t} = \frac{e^t}{1 + e^t}, \text{ and}$$

$$u_2' = \frac{y_1 g(t)}{W} = \frac{e^{-t}}{-e^{-3t}} \frac{1}{1+e^t} = -\frac{e^{2t}}{1+e^t}.$$

To do both of these integrals, we use the substitution $u = 1+e^t$, $du = e^t dt$.
Then

$$u_1 = \int \frac{du}{u} = \ln u = \ln(1+e^t);$$

$$u_2 = \int \frac{-(u-1)}{u} du = \ln u - u = \ln(1+e^t) - (1+e^t).$$

So

$$y_p = e^{-t} \ln(1+e^t) + e^{-2t} [\ln(1+e^t) - (1+e^t)],$$

and the much-sought-after general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + e^{-t} \ln(1+e^t) + e^{-2t} [\ln(1+e^t) - (1+e^t)],$$

which can be simplified, but why bother?