(A)

MAT 1322 B Winter 2016 February 10th, 8:30 Prof. Desjardins

TEST #1

Max = 15

Name: Solutions

Student Number: ____________________

- Time: 80 min.
- Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.
- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature: ____________________

(A)

1. (a) (2 points) Consider the integral $\int_1^4 \frac{3}{(x-2)^{5/3}}\,dx$. Does it converge or diverge? If it converges, give its value.

$$\int_1^4 \frac{3}{(x-2)^{5/3}}\,dx = \int_1^2 \frac{3}{(x-2)^{5/3}}\,dx + \int_2^4 \frac{3}{(x-2)^{5/3}}\,dx$$

where $\int_1^2 \frac{3}{(x-2)^{5/3}}\,dx = \lim_{t\to 2^-} \int_1^t \frac{3}{(x-2)^{5/3}}\,dx$

$$= \lim_{t\to 2^-} 3\left(-\frac{3}{2}\right)(x-2)^{-2/3}\Big|_1^t$$

$$= \lim_{t\to 2^-} -\frac{9}{2}\left[(t-2)^{-2/3} - 1\right]$$

$$= -\infty \quad (\text{so diverges})$$

$\therefore \int_1^4 \frac{3}{(x-2)^{5/3}}\,dx$ $\boxed{\text{diverges}}$

(b) (1 point) Use the Comparison Test to determine if the integral $\int_2^\infty \frac{3+\sin x}{x^3+5x}\,dx$ converges or diverges.

for all x $\quad -1 \le \sin x \le 1$ so $\quad 2 \le 3 + \sin x \le 4$

and if $x \geq 2$, $\quad x^3 + 5x > x^3$, so $\quad \frac{1}{x^3+5x} < \frac{1}{x^3}$

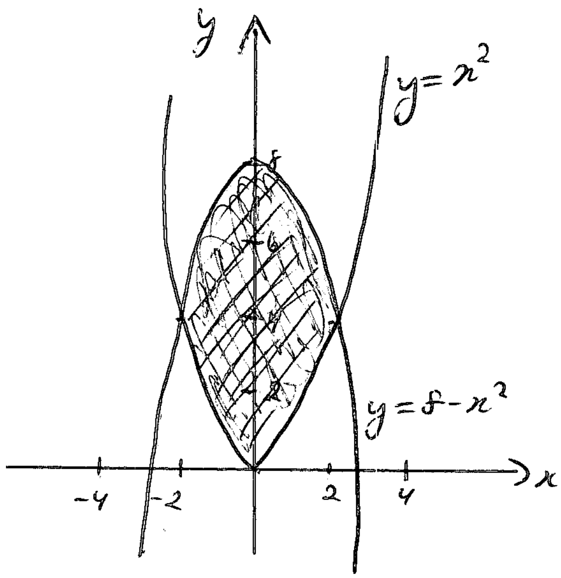
thus $\quad \frac{3+\sin x}{x^3+5x} \le \frac{4}{x^3+5x} < \frac{4}{x^3}$

and so $\quad \int_2^\infty \frac{3+\sin x}{x^3+5x}\,dx < \int_2^\infty \frac{4}{x^3}\,dx$ which is known to converge ($p = 3 > 1$)

$\therefore \int_2^\infty \frac{3+\sin x}{x^3+5x}\,dx$ $\boxed{\text{converges}}$

(A)

2. (3 points) Sketch the region bounded by the curves $y = x^2$ and $y = 8 - x^2$. What is the area of the region?



points of intersection:

$$x^2 = 8 - x^2$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = \pm 2$$

the area is

$$A = \int_{-2}^{2} ((8 - x^2) - x^2)\,dx$$

$$= \int_{-2}^{2} (8 - 2x^2)\,dx$$

$$= 2\int_{0}^{2} (8 - 2x^2)\,dx \quad \text{(by symmetry)}$$

$$= 2\left(8x - \frac{2}{3}x^3 \Big|_0^2\right)$$

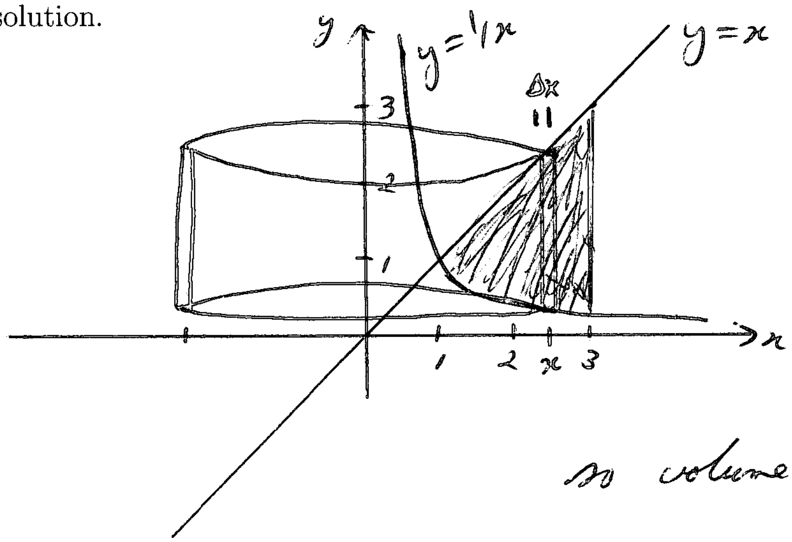
$$= 2\left(16 - \frac{16}{3} - 0\right)$$

$$= 2\left(32/3\right)$$

$$= \boxed{64/3} \simeq \boxed{21.33}$$

(A)

3. (3 points) Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = x$, $y = 1/x$, $x = 1$ and $x = 3$ is rotated around the y-axis. Include a sketch of the region and a typical cylinder (with dimensions) in your solution.



radius of shell is x

height is $x - \frac{1}{x}$

thickness is Δx

so volume is $V = 2\pi x\left(x - \frac{1}{x}\right)\Delta x$

the volume of solid is $V = \int_1^3 2\pi x\left(x - \frac{1}{x}\right)dx$

$= 2\pi \int_1^3 (x^2 - 1)\,dx$

$= 2\pi \left(\frac{1}{3}x^3 - x \Big|_1^3\right)$

$= 2\pi \left((9-3) - \left(\frac{1}{3} - 1\right)\right)$

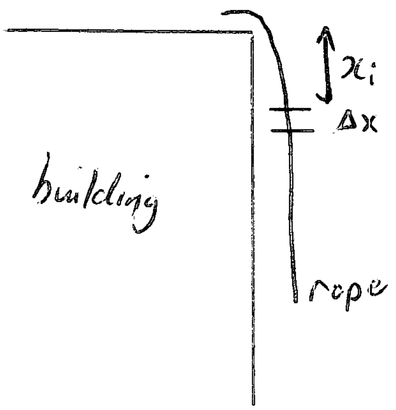
$= 2\pi \left(6 + 2/3\right)$

$= 2\pi \left(20/3\right)$

$= \boxed{40\pi/3} \simeq \boxed{41.89}$

(A)

4. (3 points) A heavy rope of length 12 m has a density of 1.5 kg/m and is hanging over the edge of a tall building. How much work is done pulling the rope to the top of the building? The acceleration of gravity is $g = 9.8\ \mathrm{m/s^2}$. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.



chop rope into lengths Δx m

consider the piece x_i m below top of building

this piece has mass $1.5\,\Delta x$ kg

and weight $1.5 g\,\Delta x = 14.7\,\Delta x$ N

this piece must be lifted x_i m, so the work done on it is $W_i = 14.7 x_i \Delta x$ J

and the total work is $W \simeq \sum_i W_i = \sum_i 14.7 x_i \Delta x$

take the limit as $\Delta x \to 0$

to get $W = \int_0^{12} 14.7 x\, dx$

$= 14.7 \left(\frac{1}{2} x^2 \Big|_0^{12} \right)$

$= 14.7 \left(\frac{1}{2} (144 - 0) \right)$

$= \boxed{1058.4\ \mathrm{J}}$

(A)

5. (a) (2 points) Use Euler's Method with step size 0.2 to estimate $y(0.4)$ where $y(x)$ is the solution of the initial value problem: $y' = 2xy\,,\ y(0) = 2\,.$

Euler's Method $y_{n+1} = y_n + h\,f(x_n, y_n)$

here $f(x,y) = 2xy$; $x_0 = 0$, $y_0 = 2$, need y_2 with $h = 0.2$

$y_1 = y_0 + h\,(2x_0 y_0) = 2 + (0.2)(2)(0)(2) = 2$

$y_2 = y_1 + h\,(2x_1 y_1) = 2 + (0.2)(2)(0.2)(2) = 2.16$

$\therefore\ y(0.4) \simeq \boxed{2.16}$

(b) (1 point) Solve the initial value problem: $(1 + x^2)\,\dfrac{dy}{dx} = 2y\,,\ \ y(0) = 3\,.$

separate the variables $\dfrac{dy}{y} = \dfrac{2}{1+x^2}\,dx$

integrate $\displaystyle\int \frac{dy}{y} = \int \frac{2}{1+x^2}\,dx + C$

we get $\ln|y| = 2\arctan x + C$

exponentiate $y = K e^{2\arctan x}$

so $y(0) = 3 \Rightarrow 3 = K e^0 \Rightarrow K = 3$

$\therefore\ \boxed{y(x) = 3e^{2\arctan x}}$