Solution to Review Questions

MAT1320, Fall 2015

1. Let $f(x) = \dfrac{3x-4}{x-2}$. What is $f^{-1}(-1)$?

Solution. $\dfrac{3x-4}{x-2} = -1$, $3x - 4 = 2 - x$. $4x = 6$, $x = 3 / 2$.

2. Solve equation $2^{2x-1} = 3^{x+1}$.

Solution. $(2x - 1)\ln 2 = (x + 1)\ln 3$, $(2\ln 2 - \ln 3)x = \ln 2 + \ln 3$. $x = \dfrac{\ln 2 + \ln 3}{2\ln 2 - \ln 3} = \dfrac{\ln 6}{\ln(4/3)}$.

3. Solve equation $\ln x - \ln (10 - x) = -3$.

Solution. $\ln \dfrac{x}{10-x} = -3$, $\dfrac{x}{10-x} = e^{-3}$. $x = e^{-3}(10 - x)$, $(1 + e^{-3})x = 10e^{-3}$. $x = \dfrac{10e^{-3}}{1+e^{-3}}$.

4. Evaluate the following expressions:

(a) $\sin(\arcsin(-0.3))$. *Answer*. -0.3.

(b) $\arccos\left(\cos\left(-\dfrac{\pi}{3}\right)\right)$. *Answer*. $\dfrac{\pi}{3}$.

(c) $\arcsin\left(\sin\left(\dfrac{2\pi}{3}\right)\right)$. *Answer*. $\dfrac{\pi}{3}$.

(d) $\sin(\arctan a)$. *Answer*. $\dfrac{a}{\sqrt{1+a^2}}$.

5. Find limits:

(a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{2+x-x^2}$.

Solution. $\lim\limits_{x\to 2} \dfrac{x^2-4}{2+x-x^2} = \lim\limits_{x\to 2} \dfrac{(x+2)(x-2)}{(2-x)(1+x)} = -\lim\limits_{x\to 2} \dfrac{x+2}{1+x} = -\dfrac{4}{3}$.

(b) $\lim\limits_{h\to 4} \dfrac{\sqrt{h+5}-3}{h-4}$.

Solution.

$$\lim_{h\to 4}\frac{\sqrt{h+5}-3}{h-4}=\lim_{h\to 4}\frac{(\sqrt{h+5}-3)(\sqrt{h+5}+3)}{(h-4)(\sqrt{h+5}+3)}=\lim_{h\to 4}\frac{h-4}{(h-4)(\sqrt{h+5}+3)}=\lim_{h\to 4}\frac{1}{\sqrt{h+5}+3}=\frac{1}{6}.$$

(c) $\lim_{h\to\infty}\frac{x^2+\sqrt{x}-1}{\sqrt{2x^4+1}}$.

Solution. $\lim_{h\to\infty}\frac{x^2+\sqrt{x}-1}{\sqrt{2x^4+1}}=\lim_{h\to\infty}\frac{(x^2+\sqrt{x}-1)/x^2}{\sqrt{2x^4+1}/x^2}=\lim_{h\to\infty}\frac{1+x^{-3/2}-1/x^2}{\sqrt{2+1/x^4}}=\frac{1}{\sqrt{2}}$.

(d) $\lim_{h\to-\infty}\frac{x}{\sqrt{4x^2+1}}$.

Solution. $\lim_{h\to-\infty}\frac{x}{\sqrt{4x^2+1}}=\lim_{h\to-\infty}\frac{x/x}{\sqrt{4x^2+1}/x}=\lim_{h\to-\infty}\frac{1}{\sqrt{4x^2+1}/(-\sqrt{x^2})}=\lim_{h\to-\infty}\frac{1}{-\sqrt{4+1/x^2}}=-\frac{1}{2}$.

6. Suppose a function is defined as

$$f(x)=\begin{cases} ax+3, & x<-2 \\ 2x+b, & -2\le x\le 3 \\ (a+1)x+2b, & x>3 \end{cases}.$$

If this function is continuous for all *x*, what are *a* and *b*?

Solution. When $x = -2$, we must have $-2a + 3 = -4 + b$, or $2a + b = 7$. When $x = 3$, we must have $6 + b = 3a + 3 + 2b$, or $3a + b = 3$. Then $a = -4$, and $b = 15$.

7. Find the derivative of the function $y = e^{\sin(x^2)}$.

Solution. By the chain rule, $y' = 2x\cos(x^2)\,e^{\sin(x^2)}$.

8. Some values of a function $y = f(x)$ and it derivative are given in the following table:

x	1	2	3	4	5
$f(x)$	2	4	5	1	3
$f'(x)$	1.2	0.5	0.2	−0.1	2.0

Let $g = f \circ f$. Fill in the following table:

x	1	2	3	4	5
$g(x)$	4	1	3	2	5
$g'(x)$	0.6	−0.05	0.4	−0.12	0.4

9. Find the second derivative of $y = \ln(x+\sqrt{x^2+4})$.

Solution. $y' = \dfrac{1}{x+\sqrt{x^2+4}}\left(1+\dfrac{2x}{2\sqrt{x^2+4}}\right) = \dfrac{1}{x+\sqrt{x^2+4}}\left(\dfrac{x+\sqrt{x^2+4}}{\sqrt{x^2+4}}\right) = \dfrac{1}{\sqrt{x^2+4}}$.

$y'' = -\dfrac{x}{(x^2+4)^{3/2}}$.

10. Find the 73rd derivative of thee function $y = \cos x$.

Solution. Since $73 = 4 \times 24 + 1$, $(\cos x)^{(73)} = (\cos x)' = -\sin x$.

11. If a function $y = f(x)$ is defined implicitly by the equation $x^3 - y^3 + x^2y + 3y = 1$. Then the derivative of this function at the point (1, 2) is

Solution. $3x^2 - 3y^2y' + 2xy + x^2y' + 3y' = 0$. $3 - 12y' + 4 + y' + 3y' = 0$. $y' = 7 / 8$.

12. Find the derivative of the function $y = \dfrac{(x^4+1)^{2/3}e^{x^2}}{\sqrt{x^2+1}}$. Do not simplify.

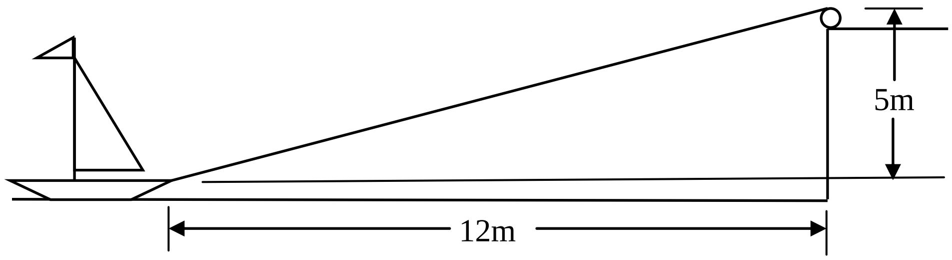
Solution. $\ln y = \dfrac{2}{3}\ln(x^4+1) + x^2 - \dfrac{1}{2}\ln(x^2+1)$. $y' / y = \dfrac{8x^3}{3(x^4+1)} + 2x - \dfrac{x}{x^2+1}$.

$y' = \dfrac{(x^4+1)^{2/3}e^{x^2}}{\sqrt{x^2+1}}\left(\dfrac{8x^3}{3(x^4+1)} + 2x - \dfrac{x}{x^2+1}\right)$.

13. Find the derivative of $y = (\sin x)^{\sin x}$.

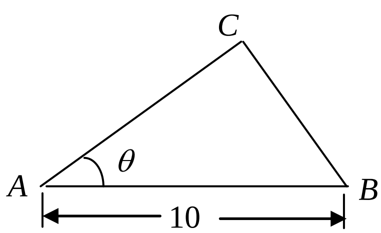
Solution. $\ln y = \sin x \ln \sin x$. $y' / y = \cos x \ln \sin x + \cos x$.
$y' = (\sin x)^{\sin x}\cos x\,(\ln \sin x + 1)$.

14. The surface of a dock is 5 meters above the deck of a boat. The boat on the water is pulled in by a cable towards the dock. When the boat is 12 meters away horizontally from the dock, it is approaching the dock horizontally at a rate of 0.5 meters per second. How fast is the cable being pulled in?



Solution. Let the length of the cable be x and the horizontal distance between the boat and the dock be y. They are both functions of time t. Then $y^2 + 25 = x^2$. Taking the derivative on both sides of this relation to t, $2yy' = 2xx'$. $x' = yy' / x$. When $y = 12$, $x = 13$ and $y' = -0.5$, $x' = 12 \times (-0.5) / 13 = -6 / 13$. The cable is pulled in at a rate 6 / 13 meter per second.

15. The hypotenuse AB of a right triangle is 10 cm. Angle CAB is increasing at a rate 0.1 radian per minute. What is the rate of change of the area of the triangle when the length of BC is 6 cm?



Solution. The area of the triangle is $A = \frac{1}{2} 10^2 \sin\theta \cos\theta = 50 \sin\theta \cos\theta$.

$A' = 50 (\cos^2\theta - \sin^2\theta)\theta'$. When $BC = 6$, $\sin\theta = 0.6$, $\cos\theta = 0.8$. Since $\theta' = 0.1$, $A' = 50 \times (0.8^2 - 0.6^2) \times 0.1 = 1.4$ cm^2 / min.

16. Use the linear approximation of the function $y = \sqrt[3]{5x+7}$ at $x = 4$ to estimate the value of $\sqrt[3]{25}$. Give the estimate as a fraction.

Solution. $y' = \dfrac{5}{3(5x+7)^{2/3}}$. When $x = 4$, $y'(4) = \dfrac{5}{27}$. The linear approximation is

$y = \dfrac{5}{27}(x-4) + 3$. Let $5x + 7 = 25$. $x = \dfrac{18}{5}$. The estimation is $\dfrac{5}{27}\left(\dfrac{18}{5} - 4\right) + 3 = \dfrac{79}{27}$.

17. If $F(x)$ is an antiderivative of the function $y = \dfrac{1}{1+x^2}$ such that $F(0) = \dfrac{2\pi}{3}$, what is $F(\sqrt{3})$?

$F(x) = \arctan x + C$. $F(0) = \frac{2\pi}{3} = C$. $F(\sqrt{3}) = \frac{\pi}{3} + \frac{2\pi}{3} = \pi$.

18. Suppose $\int_1^3 f(x)dx = 5$, $\int_2^4 f(x)dx = 9$, $\int_1^4 f(x)dx = 11$. Find $\int_2^3 (5f(x) - 2x - 3)dx$.

Solution. $\int_2^3 (5f(x) - 2x - 3)dx = 5\int_2^3 f(x)dx - \int_2^3 (2x+3)dx$.

Since $\int_1^3 f(x)dx + \int_2^4 f(x)dx = \int_1^4 f(x)dx + \int_2^3 f(x)dx$, $\int_2^3 f(x)dx = 5 + 9 - 11 = 3$.

$\int_2^3 (2x+3)dx = \left[x^2 + 3x\right]_{x=2}^3 = 8$. $\int_2^3 (5f(x) - 2x - 3)dx = 15 - 8 = 7$.

19. Suppose some values of a function $y = f(x)$ is listed in the following table:

x	1	1.2	1.4	1.6	1.8	2	2.2
y	1.2	1.8	2.1	2.2	1.9	1.7	1.4

Use all the values in the table and the left sum, right sum, trapezoidal rule, and Simpson's rule, respectively, to estimate the definite integral $\int_1^{2.2} f(x)dx$.

Solution. $n = 6$, $h = 0.2$.

$L_6 = 0.2 \times (1.2 + 1.8 + 2.1 + 2.2 + 1.9 + 1.7) = 2.180$.
$R_6 = 0.2 \times (1.8 + 2.1 + 2.2 + 1.9 + 1.7 + 1.4) = 2.220$.
$T_6 = \frac{0.2}{2} \times (1.2 + 2 \times 1.8 + 2 \times 2.1 + 2 \times 2.2 + 2 \times 1.9 + 2 \times 1.7 + 1.4) = 2.200$.
$S_6 = \frac{0.2}{3} \times (1.2 + 4 \times 1.8 + 2 \times 2.1 + 4 \times 2.2 + 2 \times 1.9 + 4 \times 1.7 + 1.4) = 2.227$.

20. Use the midpoint rule with $n = 4$ to estimate the definite integral $\int_0^1 e^{-x^2} dx$.

Solution. $n = 4$, $h = 0.25$. Interval [0, 1] is subdivided by points 0.25, 0.5, and 0.75 into four subintervals. Calculate the values of function e^{-x^2} at points $x = 0.125$, 0.375, 0.625, and 0.875:

x	0	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
y		1.016		1.151		1.478		2.150	

$M_6 = 0.25 \times (1.016 + 1.151 + 1.478 + 2.150) = 1.471$.

21. Suppose a particle is moving along the x-axis with velocity (in m/sec) $v = 100 - t^2$, $t \geq 0$. Find the total **distance** (not the displacement!) it travels from $t = 0$ to $t = 12$.

Solution. This particle moves forward when $t < 10$, and it moves backward when $t > 10$. The distance it moves forward is $\int_0^{10}(100-t^2)dt = \left[100t-\frac{t^3}{3}\right]_{t=0}^{10} = \frac{2000}{3}$, and the distance it moves backward is $--\int_{10}^{12}(100-t^2)dt = -\left[100t-\frac{t^3}{3}\right]_{t=10}^{12} = \frac{128}{3}$. The total distance it traversed is $\frac{2128}{3}$ meters.

22. If $F(x) = \int_{x^2}^{x^3}\sqrt{2t^2+1}dt$, find $F'(x)$.

Solution. $F(x) = \int_0^{x^3}(2t^2+1)^{1/3}dt - \int_0^{x^2}(2t^2+1)^{1/3}dt$. $F'(x) = 3x^2(2x^6+1)^{1/3} - 2x(2x^4+1)^{1/3}$.

23. Evaluate the definite integral $\int_1^4 \frac{x^2-1}{\sqrt{x}}dx$.

Solution. $\int_1^4 \frac{x^2-1}{\sqrt{x}}dx = \int_1^4 (x^{3/2}-x^{-1/2})dx = \left[\frac{2}{5}x^{5/2}-2x^{1/2}\right]_{x=1}^{4} = \frac{52}{5}$.

24. Evaluate the definite integral $\int_0^2 \frac{x^3}{\sqrt{2x^2+1}}dx$.

Solution. Let $u = 2x^2 + 1$. $u' = 4x$. When $x = 0$, $u = 1$; when $x = 2$, $u = 9$.

$$\int_0^2 \frac{x^3}{\sqrt{2x^2+1}}dx = \int_1^9 \frac{x^3}{\sqrt{2x^2+1}}\left(\frac{1}{4x}\right)du = \frac{1}{8}\int_1^9 \frac{u-1}{\sqrt{u}}du = \frac{1}{8}\left[\frac{2}{3}u^{3/2}-2u^{1/2}\right]_{u=1}^{9} = \frac{5}{3}.$$

25. Calculate the indefinite integral $\int x^2 \arctan x dx$.

Solution. $\int x^2 \arctan x dx = \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx$

$= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 - \frac{1}{6}\ln(1+x^2) + C$.

26. Calculate the definite integral $\int \frac{2x+1}{3x^2-2x-1}dx$.

Solution. $\frac{2x+1}{3x^2-2x-1} = \frac{A}{3x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(3x+1)}{(3x+1)(x-1)}$. $A = -\frac{1}{4}$, $B = \frac{3}{4}$.

$$\int \frac{2x+1}{3x^2-2x-1}dx = -\frac{1}{4}\int \frac{1}{3x+1}dx + \frac{3}{4}\int \frac{1}{x-1}dx = -\frac{1}{12}\ln|3x+1| + \frac{3}{4}\ln|x-1| + C.$$

27. Calculate indefinite integral $\int \frac{x+2}{x^2+2x+5}dx$.

Solution. $x^2 + 2x + 5 = (x + 1)^2 + 4$. Let $u = \frac{1}{2}(x + 1)$. Then $u' = \frac{1}{2}$, $x^2 + 2x + 5 = 4(u^2 + 1)$, and $x + 2 = 2u + 1$.

$$\int \frac{x+2}{x^2+2x+5}dx = \int \frac{x+2}{x^2+2x+5}(2)\,du = \int \frac{2(2u+1)}{4(u^2+1)}du = \int \frac{u}{u^2+1}du + \frac{1}{2}\int \frac{1}{u^2+1}du$$
$$= \frac{1}{2}\ln(u^2+1) + \frac{1}{2}\arctan u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$

28. Calculate the indefinite integral $\int \frac{x+1}{x(x^2+2x+5)}dx$.

Solution. $\frac{1}{x(x^2+2x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5} = \frac{(A+B)x^2+(2A+C)x+5A}{x(x^2+2x+5)}$.

Then $5A = 1$, $A = 1/5$. $A + B = 0$, $B = -1/5$, $2A + C = 0$, $C = -2/5$.

$$\int \frac{1}{x(x^2+2x+5)}dx = \frac{1}{5}\int \frac{1}{x}dx - \frac{1}{5}\int \frac{x+2}{x^2+2x+5}dx = \frac{1}{5}\ln|x| - \frac{1}{10}\ln(x^2+2x+5) - \frac{1}{10}\arctan\left(\frac{x+1}{2}\right) + C.$$

29. Calculate the indefinite integral $\int \frac{1}{(1-x^2)^{3/2}}dx$.

Solution. Let $x = \sin u$, $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $x' = \cos u$, $(1 - x^2)^{3/2} = \cos^3 u$.

$$\int \frac{1}{(1-x^2)^{3/2}}dx = \int \frac{1}{\cos^3 u}\cos u\,du = \tan u + C = \frac{x}{\sqrt{1-x^2}} + C.$$

30. Consider function $y = \frac{x^{1/5}}{x+1}$.

(a) Find the first and the second derivatives of this function.

(b) Find critical numbers of this function.

(c) For which values of x is this function increasing / decreasing?

(d) Find all local max / min of this function, if any.

(e) For which values of x is the graph of this function concave up / down?

(f) Find all inflection points, if any.

(g) Find all vertical/ horizontal asymptotes, if any.

(h) Sketch the graph of this function.

Solution. (a) $y' = -\dfrac{4x-1}{5x^{4/5}(x+1)^2}$, $y'' = \dfrac{2(18x^2-9x-2)}{25x^{9/5}(x+1)^3}$.

(b) Critical numbers are $x = 0$, $x = 1/4$. $x = -1$ is not a critical number because it is not in the domain of the function.

(c) $y' > 0$ when $x < -1$ or $-1 < x < 0$ or $0 < x < 1/4$, and $y' < 0$ when $x > 1/4$. The graph has a vertical tangent line at $x = 0$. The function is increasing when $x < -1$ or $-1 < x < 1/4$, and it is decreasing when $x > 1/4$.

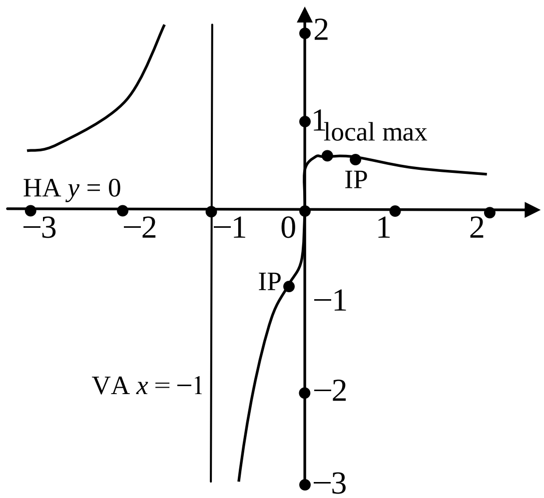
(d) Function y attains a local maximum at $x = 1/4$. $y(1/4) = \dfrac{2^{8/5}}{5}$. No local minimum.

(e) Let $y'' = 0$. $18x^2 - 9x - 2 = 0$, $x = -\dfrac{1}{6}, \dfrac{2}{3}$. $y'' > 0$ when $x < -1$ or $-\dfrac{1}{6} < x < 0$ or $x > \dfrac{2}{3}$, and $y'' < 0$ when $-1 < x < -\dfrac{1}{6}$ or $x > \dfrac{2}{3}$. The graph of the function is concave up when $x < -1$ or $-\dfrac{1}{6} < x < 0$ or $x > \dfrac{2}{3}$, and it is concave down when $-1 < x < -\dfrac{1}{6}$ or $x > \dfrac{2}{3}$.

(f) Inflection points are $\left(-\dfrac{1}{6}, -\dfrac{6^{4/5}}{5}\right)$, $\left(\dfrac{2}{3}, \dfrac{2^{1/5}\cdot 3^{4/5}}{5}\right)$.

(g) Vertical asymptote: $x = -1$, horizontal asymptote $y = 0$.

(h) The graph:



31. Find $\lim_{x\to 0}\dfrac{x-\sin x}{x-\tan x}$.

Solution. $\lim_{x\to 0}\dfrac{x-\sin x}{x-\tan x}=\lim_{x\to 0}\dfrac{1-\cos x}{1-\sec^2 x}=\lim_{x\to 0}\dfrac{\sin x}{-2\sec x(\sec x\tan x)}=\lim_{x\to 0}\dfrac{\sin x}{-2(\sin x/\cos^3 x)}$

$=-\dfrac{1}{2}\lim_{x\to 0}\cos^3 x=-\dfrac{1}{2}$.

32. Find $\lim_{x\to\pi/2}\left(x-\dfrac{\pi}{2}\right)\tan x$.

Solution. $\lim_{x\to\pi/2}\left(x-\dfrac{\pi}{2}\right)\tan x=\lim_{x\to\pi/2}\dfrac{x-\pi/2}{\cot x}=\lim_{x\to\pi/2}\dfrac{1}{\sec^2 x}=0$.

33. Find $\lim_{x\to 0}(1-2x)^{1/x}$.

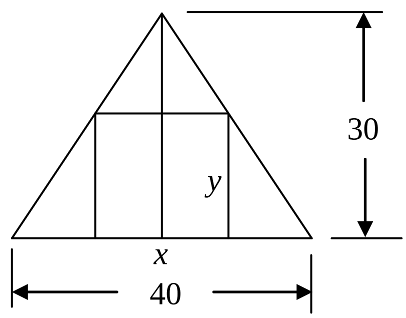
Solution. Let $y=(1-2x)^{1/x}$. Then $\ln y=\dfrac{\ln(1-2x)}{x}$. $\lim_{x\to 0}\ln y=\lim_{x\to 0}\dfrac{\ln(1-2x)}{x}=\lim_{x\to 0}\dfrac{-2}{1-2x}=-2$.

Hence, $\lim_{x\to 0}(1-2x)^{1/x}=e^{-2}$.

34. Find $\lim_{x\to 1^+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$.

Solution. $\lim_{x\to 1^+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)=\lim_{x\to 1^+}\left(\dfrac{x\ln x-x+1}{(x-1)\ln x}\right)=\lim_{x\to 1^+}\left(\dfrac{\ln x}{1-1/x+\ln x}\right)=\lim_{x\to 1^+}\left(\dfrac{1/x}{1/x^2+1/x}\right)=\dfrac{1}{2}$.

35. Find the maximum area of a rectangle inscribed in a isosceles triangle with base 40 cm and height 30 cm. Justify that what you got is an absolute maximum.



Solution. Let the height of the rectangle be y and the width be x. Then the area $A = xy$. Since

$\frac{y}{30} = \frac{20 - x/2}{20}$, $y = 30\left(1 - \frac{x}{40}\right)$. $A = 30x\left(1 - \frac{x}{40}\right) = 30x - \frac{3x^2}{4}$, $0 \le x \le 40$. Let $A' = 30 - \frac{3}{2}x = 0$, $x = 20$. Then $y = 15$, and $A = 300$. $A' > 0$ when $x < 20$ and $A' < 0$ when $x > 20$. Function $A(x)$ attains a local maximum at $x = 20$. Since $A(0) = A(40) = 0$, this local maximum is an absolute maximum.

36. A window with perimeter 10 meters has the shape of a rectangle surmounted by an equilateral triangle. Find the dimensions of the window so that the area of the window is maximized.

Solution. Let h be the height, and w be the width of the rectangle part of the window. Then the area of the window is $A = wh + \frac{\sqrt{3}}{4}w^2$. The perimeter of the window is $3w + 2h = 10$.

$h = \frac{1}{2}(10 - 3w)$. Hence, $A = \frac{1}{2}w(10 - 3w) + \frac{\sqrt{3}}{4}w^2$, $0 \le w \le \frac{10}{3}$.

Let $A' = 5 - 3w + \frac{\sqrt{3}}{2}w = 5 - \frac{1}{2}(6 - \sqrt{3})w = 0$. $w = w_0 = \frac{10}{6-\sqrt{3}} \approx 2.343$, $h = \frac{1}{2}(10 - \frac{30}{6-\sqrt{3}})$ $= 5\left(1 - \frac{3}{6-\sqrt{3}}\right) \approx 1.485$. $A \approx 5.858$. $A' > 0$ when $w < w_0$, and $A' < 0$ when $w > w_0$. Function A attains a local maximum at $w = w_0$. Since $A(0) = 0$, and $A(10 / 3) = 4.811$. This local maximum is also an absolute maximum.

37. Use Newton's method to find an approximation of a root of the equation $e^x = x + 1$, with $x_1 = 2$. Stop until $| x_{n+1} - x_n | < 0.00001$.

$x_2 = 1.556684$, $x_3 = 1.327124$, $x_4 = 1.261360$, $x_5 = 1.256462$, $x_6 = 1.256431$.