## Question 1: *Score 0/1*

| Your response | Correct response |
|---|---|
| Find the arc length of the curve | Find the arc length of the curve |
| $x = \frac{t^3}{3} - 4t, \quad y = 2t^2 + 2$ | $x = \frac{t^3}{3} - 4t, \quad y = 2t^2 + 2$ |
| between $t = 0$ and $t = 3$ . | between $t = 0$ and $t = 3$ . |
| The approximate value accurate to two decimals is (0%). | The approximate value accurate to two decimals is `21±0.01`. |

Incorrect

**Total grade:** 0.0×1/1 = 0%

**Comment:**

The arc length is :

$$\int_0^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}x = \int_0^3 \sqrt{\left(t^2 - 4\right)^2 + (4t)^2}\ \mathrm{d}x$$

$$= \int_0^3 \sqrt{(t^4 - 8t^2 + 16) + 16t^2}\ \mathrm{d}x$$

$$= \int_0^3 \sqrt{t^4 + 8t^2 + 16}\ \mathrm{d}x$$

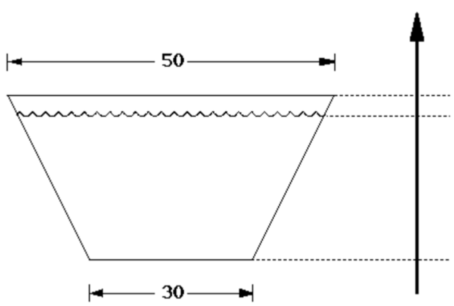
$$= \int_0^3 (t^2 + 4)\ \mathrm{d}x$$

$$= \left[ \frac{t^3}{3} + 4t \right|_0^3$$

$$\cong 21.$$

**Question 2:** *Score 0/1*

A vertical dam has the form of an isosceles trapezoid with horizontal sides parallel. The dam is $30$ m high, $30$ m in its lower part and $50$ m in its upper part. Finally, the dam retains $20$ m of water, as indicated in the figure below.



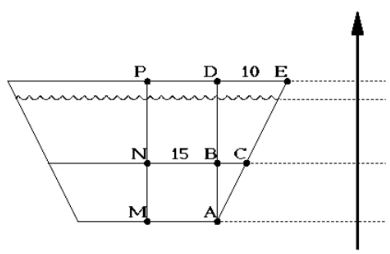
**(i)**

Let $y$ denote the height in meters measured from the base of the dam. The hydrostatic force exerted by the water on the portion of the door comprised between $y$ m and $y+\Delta y$ m is approximately $p(y)\Delta y$ $N$. What is $p(y)$ ? Note that the density of water is $\rho = 1000$ $Kg \Big/ m^3$ and the acceleration due to gravity on the earth's surface is $g = 9.8$ $m \Big/ s^2$. Express your answer as a formula.

**Your Answer:** No answer

**Correct Answer:** 9800*(20-y)*(30 + 2*(10)*y/(30))

**Comment:** The figure below shows the dam.



Given the dam has the form of an isosceles trapezoid, it is symmetric with respect to the line $MP$ that joins the midpoint $M$ of its lower horizontal side and the midpoint $P$ of its upper horizontal side. Its width at the height $y$ is $2|NC| = 2|NB| + 2|BC| = 30 + 2|BC|$. Trivially, we know the height of the dam is $|AD| = 30$ and we have $|DE| = \frac{50-30}{2} = 10$. Since the triangles $\triangle\ ABC$ and $\triangle\ ADE$ are similar and $|AB| = y$, we deduce that

$$|BC| = \frac{|AB| \cdot |DE|}{|AD|} = \frac{10y}{30}.$$

Therefore, the width of the dam at height $y$ is

$30 + 2|BC| = 30 + \frac{2\,(10y)}{30} = 30 + (2/3)y,$

and the portion of the dam comprised between $y$ m and $y + \Delta y$ m has an area of

$(30 + (2/3)y)\Delta y \quad \mathrm{m}^2.$

At height $y$, the column of water is at $(20 - y)$ meters, thus it exerts a pressure of $1000g(20 - y) = 9800(20 - y)\ N \big/ m^2$. The hydrostatic force on that portion of the dam is

$(\text{pressure}) \times (\text{area}) \cong 9{,}800(20 - y)(30 + (2/3)y)\Delta y.$

Therefore,

$$p(y) = 9{,}800(20 - y)(30 + (2/3)y).$$

**(ii)**

In Newtons, what is the total hydrostatic force exerted on the dam? Give the answer correct to $3$ significant digits.

Incorrect

**Your Answer:**
**Correct Answer:** 67,511,111.1 ± 1.0%
**Comment:** As the submerged portion of the dam is comprised between $y = 0$ and $y = 20$, the total hydrostatic force that is exerted on it is:

$$\int_0^{20} p(y)dy = 9{,}800 \int_0^{20} (20 - y)(30 + (2/3)y)\,dy$$

$$= 9{,}800\ \left[600\,y - \frac{2}{9}\,y^3 - \frac{25}{3}\,y^2\right]_0^{20} \cong 67{,}511{,}111.111111.$$

**Comments:**

## Question 3: *Score 0/1*

For which values of $k$ does the function $y = 4\cos(kt)$ satisfy the differential equation

$$9\ \frac{d^2y}{dt^2} = -16y \quad ?$$

Incorrect

List all the values in the textbox below, separating multiple entries by a semi-colon.

**Your Answer:** No answer
**Correct Answer:** 4/3; -4/3
**Comment:** By substituting $y = 4\cos(kt)$ into the differential equation, we find

$$9\ \frac{d^2}{dt^2}\ (4\cos(kt)) = -64\cos(kt)$$

which implies:

$$36k^2\ \cos(kt) = 64\cos(kt)$$

So

$$k^2 = \frac{16}{9}$$

Finally

$$k = \sqrt{\frac{16}{9}} = 4/3 \quad \text{or} \quad k = -\sqrt{\frac{16}{9}} = -4/3$$

## **Question 4:** *Score 0/1*

| Your response | Correct response |
| --- | --- |
| Consider the initial-value problem | Consider the initial-value problem |
| $y\prime = 0.2xy$ , $y(\text{-}2) = 0.3$ . | $y\prime = 0.2xy$ , $y(\text{-}2) = 0.3$ . |
| (a) Use Euler's method to estimate $y(\text{-}1)$ with step size $h = 0.5$ . | (a) Use Euler's method to estimate $y(\text{-}1)$ with step size $h = 0.5$ . |
| Give your approximation for $y(\text{-}1)$ with a precision of $\pm 0.01$ . | Give your approximation for $y(\text{-}1)$ with a precision of $\pm 0.01$ . |
| $y(\text{-}1) =$ (0%) | $y(\text{-}1) =$ `0.204±0.01` |
| (b) Use Euler's method to estimate $y(\text{-}1)$ with step size $h = 0.25$ . | (b) Use Euler's method to estimate $y(\text{-}1)$ with step size $h = 0.25$ . |
| Give your approximation for $y(\text{-}1)$ with a precision of $\pm 0.01$ . | Give your approximation for $y(\text{-}1)$ with a precision of $\pm 0.01$ . |
| $y(\text{-}1) =$ (0%) | $y(\text{-}1) =$ `0.213653±0.01` |

Incorrect

**Total grade:** 0.0×1/3 + 0.0×2/3 = 0% + 0%

**Comment:**

The differential equation we must solve is of the form $y\prime = F(x, y)$ where $F(x, y) = 0.2xy$ .

**Comment:**

The differential equation we must solve is of the form $y\prime = F(x, y)$ where $F(x, y) = 0.2xy$ .

(a) For the step size $h = 0.5$ , Euler's method consists of letting $x_0 = -2$ and $y_0 = 0.3$ (since $y(-2) = 0.3$ ), and defining recursively

$$x_{n+1} = x_n + h = x_n + 0.5$$

and

$$\begin{aligned} y_{n+1} &= y_n + F(x_n, y_n)h \\ &= y_n + (0.2x_n\, y_n)(0.5) \\ &= y_n + 0.1x_n\, y_n \\ &= y_n\,(1 + 0.1x_n)\,, \end{aligned}$$

for each $n = 0, 1, 2, \ldots$ so that we have $y(x_n) \cong y_n$ for every $n$ .

Since we want to approximate $y(-1)$ , we stop once we arrive at $x_n = -1$ .

We find:

$x_1 = -2 + 0.5 = -1.5$, $y_1 = 0.3(1 + (0.1)(-2)) = 0.24$

$x_2 = x_1 + 0.5 = -1$, $y_2 = 0.24(1 + (0.1)(-1.5)) = 0.204$

Hence the answer is $y(-1) \cong 0.204$ .

(b) For a step size of $h = 0.25$ , once again, we let $x_0 = -2$ and $y_0 = 0.3$ (since $y(-2) = 0.3$ ), and we define recursively

$$x_{n+1} = x_n + h = x_n + 0.25$$

and

$$\begin{aligned} y_{n+1} &= y_n + F(x_n, y_n)h \\ &= y_n + (0.2x_n\, y_n)(0.25) \\ &= y_n + 0.05x_n\, y_n \\ &= y_n\,(1 + 0.05x_n)\,, \end{aligned}$$

for each $n = 0, 1, 2, ...$ Such that we have $y(x_n) \cong y_n$ for every $n$ .

Since we want to approximate $y(-1)$ , we stop once we arrive at $x_n = -1$ .

We find:

$x_1 = -2 + 0.25 = -1.75$, $y_1 = 0.3(1 + (0.05)(-2)) = 0.27$

$x_2 = x_1 + 0.25 = -1.5$, $y_2 = 0.27(1 + (0.05)(-1.75)) = 0.246375$

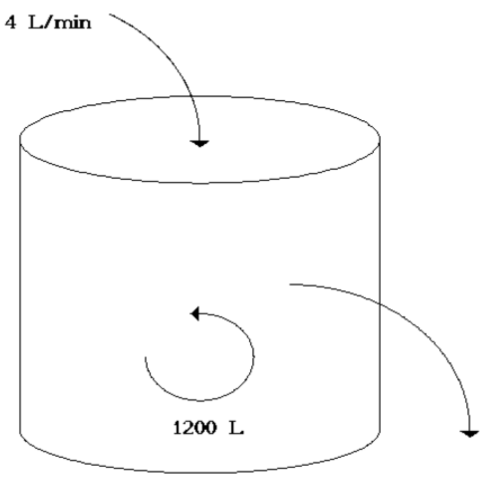
$x_3 = x_2 + 0.25 = -1.25$, $y_3 = 0.246375(1 + (0.05)(-1.5)) = 0.227897$

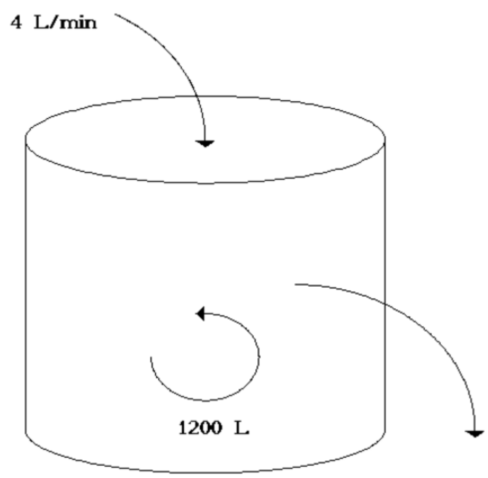
$x_4 = x_3 + 0.25 = -1$, $y_4 = 0.227897(1 + (0.05)(-1.25)) = 0.213653$

Hence the answer is $y(-1) \cong 0.213653$ .

**Question 5:** *Score 0/1*

| Your response | Correct response |
| --- | --- |




Incorrect

A $1{,}200$ L vat of beer initially contains $4$ % alcohol by volume.
Some beer containing $10$ % alcohol is then added at a rate of $4$ L/min, and the beer mixture is evacuated at the same rate.
Let $Q(t)$ be defined as the volume of alcohol present in the vat (in L) after $t$ minutes.

(i) What is the differential equation that satisfies $Q$ ?

$\frac{dQ}{dt}$ = No answer (0%) L/min

Give the correct expression in terms of $Q$ (do not write $Q(t)$ ).

(ii) Give the exact formula for $Q(t)$ as a function of $t$ .

$Q(t)$ = No answer (0%) L

(iii) What is the percentage of alcohol present in the vat after one hour?

Answer: (0%) .

Round your answer to one decimal place.

A $1{,}200$ L vat of beer initially contains $4$ % alcohol by volume.
Some beer containing $10$ % alcohol is then added at a rate of $4$ L/min, and the beer mixture is evacuated at the same rate.
Let $Q(t)$ be defined as the volume of alcohol present in the vat (in L) after $t$ minutes.

(i) What is the differential equation that satisfies $Q$ ?

$\frac{dQ}{dt}$ = Correct Answer not defined L/min

Give the correct expression in terms of $Q$ (do not write $Q(t)$ ).

(ii) Give the exact formula for $Q(t)$ as a function of $t$ .

$Q(t)$ = Correct Answer not defined L

(iii) What is the percentage of alcohol present in the vat after one hour?

Answer: 5.087615±0.1 .

Round your answer to one decimal place.

**Total grade:** 0.0×2/4 + 0.0×1/4 + 0.0×1/4 = 0% + 0% + 0%

**Comment:**

(i) We know that

$$\frac{dQ}{dt}(t) = \text{(rate of alcohol entering the vat) - (rate of alcohol exiting the vat)}$$

$$= 4\left(\frac{10}{100}\right) - 4\left(\frac{Q\,(t)}{1{,}200}\right)$$

$$= \frac{120 - Q\,(t)}{300}$$

So the differential equation for $Q$ is

$$\frac{dQ}{dt} = \frac{120 - Q}{300}$$

(ii) By separation of variables, we obtain

$$\frac{dQ}{Q-120} = -\frac{dt}{300}$$

Then, integrating both sides, we find

$$\int \frac{dQ}{Q-120} = -\int \frac{dt}{300}$$

$$\Rightarrow \quad \ln|Q-120| = -\frac{t}{300} + C$$

$$\Rightarrow \quad |Q-120| = e^{-(t/300)+C}$$

$$\Rightarrow \quad Q = 120 \pm e^{-(t/300)+C}$$

and thus,
$Q = 120 + Ae^{-t/300}$ where $A = \pm e^{C}$ .

Initially, the vat contains

$Q(0) = 1{,}200\left(\frac{4}{100}\right) = 48$ liters of alcohol. Hence

$$48 = 120 + A \quad \Rightarrow \quad A = \text{-}72$$

Now with $A$ , we obtain the formula for $Q(t)$ :

$$Q(t) = 120 + (\text{-}72)e^{-t/300}$$

(iii) After one hour, we have $Q(60) = 120 + (\text{-}72)e^{-60/300} = 61.051386$ L

Therefore the percentage of alcohol is $\frac{Q\ (60)}{1{,}200}\ (100) = \frac{61.051386}{12} \cong 5.1$ %

## Question 6: *Score 0/1*

| Your response | Correct response |
|---|---|
| Solve the initial-value problem shown below: $$\frac{dy}{dx} = x\ \sqrt{1-y^2}\ , \quad y\,(0) = 0\ .$$ Give an exact formula for $y$ . $y =$ No answer (0%) | Solve the initial-value problem shown below: $$\frac{dy}{dx} = x\ \sqrt{1-y^2}\ , \quad y\,(0) = 0\ .$$ Give an exact formula for $y$ . $y =$ Correct Answer not defined |

Incorrect

**Total grade:** 0.0×1/1 = 0%

**Comment:**

After separation of variables, the differential equation becomes

$$\int \frac{dy}{\sqrt{1-y^2}} = \int x\ dx$$

We deduce

$\arcsin(y) = \frac{x^2}{2} + C$ where $C$ is a constant.

Since $y(0) = 0$ , we find $C = \arcsin(0) = 0$ , hence $\arcsin(y) = \frac{x^2}{2}$ and thus

$$y = \sin\left(\frac{x^2}{2}\right)$$

## Question 7: *Score 0/1*

| Your response | Correct response |
|---|---|
| A turkey is taken out of the oven at $14:00$ and is placed on the kitchen counter. Its initial temperature is $84\circ\mathrm{C}$ and room temperature is $20\circ\mathrm{C}$. One hour later, the turkey reaches a temperature of $57\circ\mathrm{C}$. Let $T(t)$ be the temperature of the turkey $t$ hours after the initial time. | A turkey is taken out of the oven at $14:00$ and is placed on the kitchen counter. Its initial temperature is $84\circ\mathrm{C}$ and room temperature is $20\circ\mathrm{C}$. One hour later, the turkey reaches a temperature of $57\circ\mathrm{C}$. Let $T(t)$ be the temperature of the turkey $t$ hours after the initial time. |
| (i) Supposing that the temperature of the turkey follows Newton's Law of Cooling, give the exact formula for $T(t)$, by finding and solving the corresponding differential equation. $T(t)=$ `No answer` (0%) | (i) Supposing that the temperature of the turkey follows Newton's Law of Cooling, give the exact formula for $T(t)$, by finding and solving the corresponding differential equation. $T(t)=$ `Correct Answer not defined` |
| (ii) What is the temperature of the turkey at $18:00$? Give the answer with $0.1$ precision. Answer: ___ (0%) ○ C | (ii) What is the temperature of the turkey at $18:00$? Give the answer with $0.1$ precision. Answer: `27.149357±0.1` ○ C |
| (iii) At what time will the temperature of the turkey reach $29\circ\mathrm{C}$? Give the answer as a decimal with $0.1$ precision. Answer: ___ (0%) | (iii) At what time will the temperature of the turkey reach $29\circ\mathrm{C}$? Give the answer as a decimal with $0.1$ precision. Answer: `17.6±0.1` |

Incorrect

**Total grade:** 0.0×2/5 + 0.0×1/5 + 0.0×2/5 = 0% + 0% + 0%

**Comment:**

(i) With room temperature being $20\circ\mathrm{C}$, we use Newton's Law of Cooling to obtain

$$\frac{\mathrm{d}T}{\mathrm{d}t}=k\,(T-20)$$

**Comment:**

(i) With room temperature being $20 \circ \mathrm{C}$ , we use Newton's Law of Cooling to obtain

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k\,(T-20)$$

where $k$ is a constant and $t$ represents the number of hours after $14\ :\ 00$ . By using the separation of variables method, we get

$$\begin{aligned} \int \frac{\mathrm{d}T}{T-20} &= \int k\,\mathrm{d}t \\ \Rightarrow \ln|T-20| &= kt+C \\ \Rightarrow \quad |T-20| &= e^{kt+C} \\ \Rightarrow \quad T-20 &= Ae^{kt} \quad \text{where} \quad A = \pm e^{C} \end{aligned}$$

Therefore, we obtain finally

$$T\,(t) = 20 + Ae^{kt}$$

Since $14\ :\ 00$ , represents $t=0$ , the turkey is at a temperature of $84 \circ \mathrm{C}$ . So $T\,(0) = 84$ . This gives us

$$\begin{aligned} 84 &= 20 + A \\ \Rightarrow \quad A &= 64 \\ \Rightarrow \quad T\,(t) &= 20 + 64e^{kt} \end{aligned}$$

One hour later, the turkey has reached $57 \circ \mathrm{C}$ , therefore $T\,(1) = 57$ . Thus

$$\begin{aligned} 57 &= 20 + 64e^{k} \\ e^{k} &= \frac{37}{64} \\ k &= \ln\left(\frac{37}{64}\right) \end{aligned}$$

Finally, with k we obtain $T\,(t) = 20 + 64e^{\ln(37/64)t} = 20 + 64\left(\frac{37}{64}\right)^{t}$ .

Finally, with k we obtain $T\,(t) = 20 + 64e^{\ln(37/64)t} = 20 + 64\left(\frac{37}{64}\right)^t$.

(ii) At $18\ :\ 00$, we have $t = 4$, so the temperature of the turkey is

$$T\,(4) = 27.1 \circ \mathrm{C}$$

(iii) Here we are looking for $t$ such that $T(t) = 29$, that is to say:

$$29 = 20 + 64\left(\frac{37}{64}\right)^t$$

$$\Rightarrow \quad \frac{9}{64} = \left(\frac{37}{64}\right)^t$$

$$\Rightarrow \quad \ln\left(\frac{9}{64}\right) = t\ln\left(\frac{37}{64}\right)$$

$$\Rightarrow \quad t = \frac{\ln\left(\frac{9}{64}\right)}{\ln\left(\frac{37}{64}\right)} = 3.579896$$

So the time we are looking for corresponds to $3.6$ hours after the original time, which is equivalent to $14 + 3.6 = 17.6$ hours.

## Question 8: *Score 0/1*

| Your response | Correct response |
| --- | --- |
| Biologists have established a colony of $4{,}000$ bees to inhabit St. Lawrence Island. They estimated that the island would support up to $20{,}000$ bees. Moreover, the bees' population growth rate in an unrestrained environment is estimated at $k = 0.2$ per month. Let $P(t)$ be defined as the population of bees after $t$ months. The corresponding logistic differential equation is of the form: $$\frac{dP}{dt} = f(P)$$ | Biologists have established a colony of $4{,}000$ bees to inhabit St. Lawrence Island. They estimated that the island would support up to $20{,}000$ bees. Moreover, the bees' population growth rate in an unrestrained environment is estimated at $k = 0.2$ per month. Let $P(t)$ be defined as the population of bees after $t$ months. The corresponding logistic differential equation is of the form: $$\frac{dP}{dt} = f(P)$$ |
| (i) Find $f(P)$ . $f(P) =$ No answer (0%) . Give an exact formula . | (i) Find $f(P)$ . $f(P) =$ `(0.2)*P*(1 - P/(20,000))` . Give an exact formula . |
| (ii) Knowing that the solution to the logistic equation is $$P(t) = \frac{K}{1 + Ae^{-kt}} ,$$ find the exact population after $6$ months as predicted by the model. $P(6) =$ (0%) bees. Round your answer to the nearest integer. | (ii) Knowing that the solution to the logistic equation is $$P(t) = \frac{K}{1 + Ae^{-kt}} ,$$ find the exact population after $6$ months as predicted by the model. $P(6) =$ `9,071±1` bees. Round your answer to the nearest integer. |

Incorrect

**Total grade:** 0.0×1/2 + 0.0×1/2 = 0% + 0%

**Comment:**

(i) The logistic differential equation is of the form:

$$\frac{dP}{dt} = f(P) = kP\left(1 - \frac{P}{K}\right)$$

where $k$ is the population growth rate of the bees and $K$ is the carrying capacity (maximum population) of the island. Here $k = 0.2$ per month and $K = 20{,}000$ bees, hence

**Total grade:** 0.0×1/2 + 0.0×1/2 = 0% + 0%

**Comment:**

(i) The logistic differential equation is of the form:

$$\frac{dP}{dt} = f(P) = kP\left(1 - \frac{P}{K}\right)$$

where $k$ is the population growth rate of the bees and $K$ is the carrying capacity (maximum population) of the island. Here $k = 0.2$ per month and $K = 20{,}000$ bees, hence

$$f(P) = 0.2P\,(1 - \frac{P}{20{,}000})$$

(ii) We know that

$$P(t) = \frac{20{,}000}{1 + Ae^{-0.2t}}$$

To determine $A$, we use $P(0) = 4{,}000$

$\Rightarrow \quad 4{,}000 = \frac{20{,}000}{1+A}$

$\Rightarrow \quad A = \frac{20{,}000}{4{,}000} - 1 = 4$

Therefore

$$P(t) = \frac{20{,}000}{1 + 4e^{-0.2t}}$$

and

$$P(6) = \frac{20{,}000}{1 + 4e^{-0.2(6)}} \quad \cong \quad 9{,}071$$

After $6$ months, the model predicts a population of $9{,}071$ bees.