MAT 2324, Differential Equations
Prof. Frithjof Lutscher, University of Ottawa
Winter 2016, Assignment #3
Due Tuesday February 9

Problem 1: A lake with volume V km^3 is fed with clear water by a river at a rate of r km^3/ per year. At the banks of the lake, an industrial compound releases a pollutant into the lake at a rate of p km^3/ per year. The lake feeds another river, and the outflow is such that the volume of the lake remains constant. Assume that the lake is well mixed so that the concentration of the pollutant is uniform. Denote by $x(t)$ the volume of the pollutant in the lake at time t and by ρ its mass density measured in kg/km^3. Then the total mass of the pollutant is $Q(t) = \rho x(t)$.

(a) Using the mass-balance equation that the change of Q is given by the total inflow minus the total outflow, write down the differential equation for $x(t)$.

(b) Denote by $c(t) = x(t)/V$ the concentration of the pollutant (in %), assume $c(0) = 0$ and write the inivial value problem for $c(t)$.

(c) Solve the IVP in part (b).

(d) It has been found that a pollutant concentration above 2% is damaging to a fish population in the lake. Assume that $V = 100$, $r = 50$ and $p = 2$, how long will it take for the concentration to reach this threshold?

(e) How long would it take for $p = 1$?

Problem 2: Consider the ODE

$$\frac{dy}{dt} = (y-2)(\lambda - (y-2)^2 + 3)$$

With parameter $\lambda \in \mathbb{R}$.

(a) Draw a phase-line diagram for $\lambda > -3$. Indicate the steady states and their stability.

(b) Do the same as in (a) but for $\lambda < -3$.

(c) Do the same as in (a) but for $\lambda = -3$..

(d) Summarize your results in a bifurcation diagram.

Problem 3: The value $P(t)$ of a savings plan at time t changes according to interest paid and deposits made.

$$\begin{matrix}\text{rate of change} \\ \text{of } P(t)\end{matrix} = \begin{matrix}\text{rate of change} \\ \text{due to interest}\end{matrix} + \begin{matrix}\text{rate of change} \\ \text{due to deposits}\end{matrix}$$

Assuming that interest is compounded continuously (with annual interest rate r) and deposits are made continuously at a rate of $D(t)$ dollar per year, we obtain the ODE

$$\frac{dP}{dt} = rP + D(t).$$

Since income is typically lower at the beginning of a career than towards the end, we assume that the annual savings increase with time as

$$D(t) = 10000(1 - e^{-\gamma t}) \text{ dollars per year}$$

where $\gamma > 0$ is constant.

(a) Solve the IVP $\frac{dP}{dt} = rP + 10000(1 - e^{-\gamma t})$, $P(0) = 0$. Your answer should contain parameters r and γ.

(b) Assume that you would like to have 1 million dollar when you retire. Assume as well that you can make deposits according to the function D with $\gamma = 0.1$ and that the interest rate is $r = 10\%$. How long will you have to work until you can retire? At the time of retirement, how large will your annual deposit rate, $D(t)$, be?

NOTE: Additional (up to) 2 points will be given for the presentation of your answers (clear logic, legible riting and correct English).