Sample Problems for Final

Calculus 1 MAT 1320 B

1. DIFFERENTIATION TECHNIQUES

a) Find the derivatives of the following functions:

(i) $y = 3x^{-2} + \cos(x) - e$

Solution:

$$y' = -6x^{-3} - \sin(x)$$

(ii) $y = \sin(x^2)(3x^2 - x^3)$

Solution:

$$y' = 2x\cos\left(x^2\right)\left(3x^2 - x^3\right) + \sin\left(x^2\right)\left(6x - 3x^2\right)$$

(iii) $y = e^{2x} + \ln(x^3)(x+1)$;

Solution:

$$y' = 2e^{2x} + \frac{3}{x}(x+1) + \ln\left(x^3\right)$$

(iv) $y = \cos(x^2+1)\sin(x^2-1)$;

Solution:

$$y' = -2x\sin\left(x^2+1\right)\sin\left(x^2-1\right) + 2x\cos\left(x^2+1\right)\cos\left(x^2-1\right)$$

(v) $y = \ln\left((x-1)^3\right)$

Solution:

$$y' = \frac{3}{x-1}$$

(vi) $y = \arcsin\left(\log_2\left(x + x^2\right)\right)$.

Solution:

$$y' = \left(\frac{1}{\sqrt{1 - \left(\log_2\left(x + x^2\right)\right)^2}}\right)\left(\frac{1 + 2x}{\ln(2)(x + x^2)}\right).$$

b) Find $\frac{dy}{dx}$.

(i) $\cos(y) + x^2 y = y + x^3 - 2;$

Solution:

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 - \sin(y) - 1}$$

(ii) $x^3 y - xy^3 + xy = y^2 x - yx^2 - yx$

Solution:

$$\frac{dy}{dx} = \frac{y^3 + y^2 - 2y - 2xy - 3x^2 y}{x^3 + x^2 + 2x - 3xy^2 - 2xy}$$

(iii) $y = \sin(x)^{x^2 - 1}$

Solution:

$$y' = \left(2x\,\ln(\sin(x)) + \frac{(x^2 - 1)\cos(x)}{\sin(x)}\right)\sin(x)^{x^2 - 1}$$

(iv) $y = x^{\ln(x) + x^3}$

Solution:

$$y' = \left(\left(\frac{1}{x} + 3x^2\right)\ln(x) + \left(\ln(x) + x^3\right)\frac{1}{x}\right)x^{\ln(x) + x^3}$$

2. APPLICATIONS AND INTERPRETATIONS OF DERIVATIVES

a) Find an equation of the tangent line to the following curves at the given points.

(i) $f(x) = \ln\left((x-1)^3\right)$ at $x = 2$;

Solution: We have that

$$f'(2) = 3, \qquad f(2) = 0.$$

Thus,

$$y = 3(x-2).$$

is an equation of the tangent line.

(ii) $1 + y^2 - xy = e^y - x$ at $(0,0)$.

Solution: We have that

$$y' = \frac{y-1}{2y - x - e^y},$$

so

$$\left.\frac{dy}{dx}\right|_{(0,0)} = 1.$$

Thus, an equation of the tangent line is

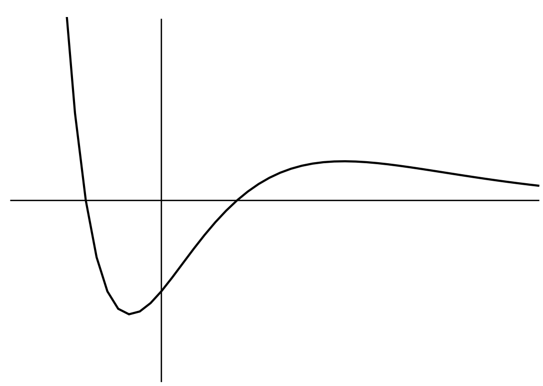
$$y = x.$$

b) Sketch the graph of the following functions: find the domain, intercepts, asymptotes, intervals where the function is increasing/decreasing, intervals of concavity.

(i) $y = e^{-x}(x^2 - 1)$

Solution:

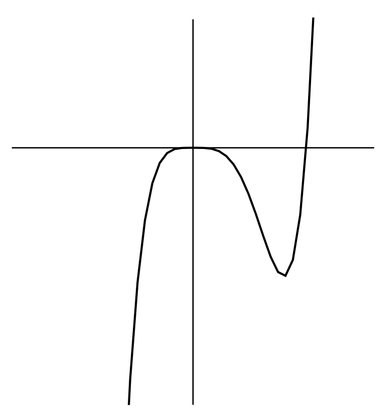
- Domain: all real numbers.
- y-intercept: -1; x-intercept: $\pm$ 1.
- f increasing on $(1 - \sqrt{2}, 1 + \sqrt{2})$.
- f decreasing on $(-\infty, 1 - \sqrt{2})$ and $(1 + \sqrt{2}, \infty)$.
- f concave up on $(-\infty, 2 - \sqrt{3})$ and $2 + \sqrt{3}, \infty)$.
- f concave down on $(2 - \sqrt{3}, 2 + \sqrt{3})$.



(ii) $y = 4x^5 - 5x^4$

Solution:

- Domain: all real numbers.
- y-intercept: 0; x-intercept: $\frac{5}{4}$.
- f increasing on $(-\infty, 0)$ and $(1, \infty)$.
- f decreasing on $(0, 1)$.
- f concave up on $(3/4, \infty)$.
- f concave down on $(-\infty, 3/4)$.



c) Using L'Hopital's rule obtain $\lim_{x\to\infty} x^{\tan(1/x)}$.

Solution: First consider

$$\lim_{x\to\infty} x^{\tan(1/x)} = \lim_{x\to\infty} e^{\ln\left(x^{\tan(1/x)}\right)} = \lim_{x\to\infty} e^{\tan(1/x)\ \ln(x)}$$

Set $u = \tan(1/x)\ \ln(x)$. Since

$$\begin{aligned}
\lim_{x\to\infty} \tan(1/x)\ \ln(x) \quad &= \quad \lim_{x\to\infty} \frac{\ln(x)}{(\tan(1/x))^{-1}} \\
&\underset{L'H}{=} \quad \lim_{x\to\infty} \frac{1/x}{\frac{\sec^2(1/x)}{\tan^2(1/x)x^2}} = \lim_{x\to\infty} \sin^2(1/x)x = \lim_{x\to\infty} \frac{\sin^2(1/x)}{1/x} \\
&\underset{L'H}{=} \quad \lim_{x\to\infty} 2\sin(1/x)\cos(1/x) = 0.
\end{aligned}$$

Thus,

$$\lim_{x\to\infty} x^{\tan(1/x)} = \lim_{x\to\infty} e^{\tan(1/x)\ \ln(x)} = \lim_{u\to 0} e^0 = 1.$$

d) Consider the function $f(x) = \sqrt{1+x}$. Find its linear approximation at $a = 2$. Then, estimate $\sqrt{2.8}$.

Solution: Since

$$f'(x) = \frac{1}{2\sqrt{1+x}},$$

we have that

$$f'(2) = \frac{1}{2\sqrt{3}}, \qquad\qquad f(2) = \sqrt{3}.$$

Thus, the linear approximation is

$$L_2(x) = \frac{1}{2\sqrt{3}}(x-2) + \sqrt{3}.$$

Now, since $\sqrt{2.8} = \sqrt{1+x}$ when $x = 1.8$, we have then

$$\sqrt{2.8} \approx L_2(1.8) = -\frac{0.2}{2\sqrt{3}} + \sqrt{3}.$$

e) Two ships, A and B, leave Vancouver together and sail due west. Ship A sails at 20 miles per hour and ship B sails at 25 miles per hour. Ten miles out to sea, A turns due north and B continues due west. How fast ae they moving away from each other 4 hours after departing?

Solution: Ship A moves at a rate of 20 miles per hour, so it reaches ten miles after half an hour. Thus, after four hours of departure, ship A has moved due north for 3 hours and a half, which accounts for 70 miles.

Now, ship B moves at a rate of 25 miles per hour, so after four hours of departure it has moved 100 miles, 90 miles more than ship A did on that direction.

Now, let t be the time in hours. Let $x(t)$ be the distance, in miles, travelled by ship A after the turning point. In particular we have that

$$\frac{dx}{dt} = 20.$$

Let $y(t)$ be the distance, in miles, travelled by ship B after the turning point of ship A. In particular we have that

$$\frac{dy}{dt} = 25.$$

Let $z(t)$ be the distance, in miles, between the two ships. Hence

$$z^2(t) = x^2(t) + y^2(t).$$

So, after four hours

$$z(4) = \sqrt{90^2 + 70^2} = \sqrt{13000}.$$

We want $z'(4)$. From above we have

$$\begin{aligned} z^2(t) &= x^2(t) + y^2(t) \\ 2z(t)z'(z) &= 2x(t)x'(t) + 2y(t)y'(t) \\ z'(t) &= \frac{x(t)x'(t) + y(t)y'(t)}{z(t)} \\ z'(t) &= \frac{20x(t) + 25y(t)}{z(t)} \end{aligned}$$

Thus,

$$z'(4) = \frac{20(70) + 25(90)}{\sqrt{13000}}$$

f) A window in the shape of a rectangle with a semicircle on the top is to be made with a perimeter of 4 meters. What is the largest possible area for such a window?

Solution: Let x be the base of the window, so it is also the diameter of the semicircle. Let y be the height of the rectangle. We have then

$$4 = 2x + 2y + x\pi$$

Thus

$$y = \frac{4 - x(2+\pi)}{2}.$$

Now, the area of the window is

$$A = yx + \frac{x^2\pi}{4} = x\left(\frac{4 - x(2+\pi)}{2}\right) + \frac{x^2\pi}{4}$$

Its derivative is

$$A' = \frac{4 - 4x - 2x\pi}{2} + \frac{2\pi x}{4}$$

So the solution of $0 = A'(x)$ is $x = 4/(4+\pi)$.

3. INTEGRALS

a) An object on the x-axis has velocity $v(t) = 2t - t^2$ at tme t. If it starts out at $x = -1$ at time $t = 0$, where is it at time $t = 3$? How far has it travelled?

Solution: Let $x = F(t)$ be the position of the object at time t. Its displacement from $t = 0$ to $t = 3$ is

$$\int_0^3 2t - t^2 \, dt = \left(t^2 - \frac{t^3}{3}\right) |_0^3 = 0.$$

This means that at $t = 3$ the object is again at $x = -1$.

In order to answer how far it has travelled, notice that the object turns around as v changes sign. In other words, when $v(t) = 0$ is when the object changes direction: for $0 < t < 2$ we have that $v(t) > 0$, and for $2 < t < 3$ we have that $v(t) < 0$. Thus, the total distance travelled is

$$\int_0^2 2t - t^2 \, dt - \int_2^3 2t - t^2 \, dt = \frac{8}{3}.$$

b) Obtain $\frac{d}{dx}\int_a^{\sin(x)} \arctan(\ln(t)-1)\,dt.$

Solution:

$$\frac{d}{dx}\int_a^{\sin(x)} \arctan(\ln(t)-1)\,dt = \arctan\left(\ln(\sin(x))-1\right)\cos(x).$$

c) Find $f(x)$ if $f'(x) = xe^x$ and $f(0) = 1$.

Solution: Since

$$\int xe^x\,dx = xe^x - e^x + C$$

we have that

$$f(x) = xe^x - e^x + C$$

Now, from

$$1 = f(0) = 0e^0 - e^0 + C = -1 + C$$

we conclude

$$f(x) = xe^x - e^x + 2.$$

d) Find

(i) $\int \sin^7(x)\,dx.$

Solution: Particular Case, Trigonometric Function: First

$$\begin{aligned}
\int \sin^7(x)\,dx &= \int \sin(x)\cos^6(x)\,dx = \int \sin(x)(1-\cos^2(x))^3\,dx \\
&= \int \sin(x)\left(1 - 3\cos^2(x) + 3\cos^4(x) - \cos^6(x)\right)\,dx \\
&= \int \sin(x)\,dx - 3\int \sin(x)\cos^2(x)\,dx + 3\int \sin(x)\cos^4(x)\,dx - \int \sin(x)\cos^6(x)\,dx
\end{aligned}$$

Taking $u = \cos(x)$, we have $du = -\sin(x)\,dx$, so we get

$$\begin{aligned}
\int \sin^7(x)\,dx &= \int \sin(x)\,dx - 3\int \sin(x)\cos^2(x)\,dx + 3\int \sin(x)\cos^4(x)\,dx - \int \sin(x)\cos^6(x)\,dx \\
&= \int \sin(x)\,dx + 3\int u^2\,du - 3\int u^4\,du + \int u^6\,du \\
&= -\cos(x) + \cos^3(x) - \frac{3}{5}\cos^5(x) + \frac{1}{7}\cos^7(x) + C.
\end{aligned}$$

(ii) $\displaystyle\int \frac{x^3}{\sqrt{x^4+2}}\, dx.$

Solution: Proceed by Substitution: take $u = x^4 + 2$, so $du = 4x^3\, dx$. Then

$$\int \frac{x^3}{\sqrt{x^4+2}}\, dx = \frac{1}{4}\int u^{-1/2}\, du = \frac{\sqrt{x^4+2}}{2} + C.$$

(iii) $\displaystyle\int \frac{\sin(x)\, \ln(\cos(x))}{\cos(x)}\, dx.$

Solution: Proceed by Substitution: Take $u = \ln(\cos(x))$, so $du = \dfrac{-\sin(x)}{\cos(x)}\, dx$. Then

$$\int \frac{\sin(x)\, \ln(\cos(x))}{\cos(x)}\, dx = -\int u\, du = -\frac{\ln(\cos(x))}{2} + C.$$

(iv) $\displaystyle\int x^2\, e^{5x}\, dx.$

Solution: Proceed by Parts: First take $u = x^2$, $dv = e^{5x}\, dx$, so we have $du = 2x\, dx$ and $v = e^{5x}/5$. Then

$$\int x^2 e^{5x}\, dx = \frac{x^2 e^{5x}}{5} - \frac{2}{5}\int x e^{5x}\, dx.$$

Again by parts, with $u = x$, $dv = e^{5x}\, dx$, so we have $du = dx$ and $v = e^{5x}/5$, we have now that

$$\int x e^{5x}\, dx = \frac{x e^{5x}}{5} - \int \frac{e^{5x}}{5}\, dx = \frac{x e^{5x}}{5} - \frac{e^{5x}}{5^2}.$$

Putting all together we obtain

$$\int x^2 e^{5x}\, dx = \frac{x^2 e^{5x}}{5} - \frac{2x e^{5x}}{5^2} + \frac{2e^{5x}}{5^3} + C.$$

(v) $\displaystyle\int x \sin(3x)\, dx.$

Solution: Proceed by Parts: Take $u = x$ and $dv = \sin(3x)\, dx$, so we have $du = dx$ and $v = -\cos(3x)/3$. Then

$$\int x \sin(3x)\, dx = -\frac{x\cos(3x)}{3} + \frac{1}{3}\int \cos(3x)\, dx = -\frac{x\cos(3x)}{3} - \frac{\sin(3x)}{3^2} + C.$$

(vi) $\displaystyle\int \frac{x-1}{x^3+x^2}\,dx.$

Solution: Special Case, Partial Fractions: Since

$$\frac{x-1}{x^3+x^2} = \frac{x-1}{x^2(x+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} = \frac{A_1x(x+1)+A_2(x+1)+A_3x^2}{x^2(x+1)}$$

we should have

$$x-1 = A_1x(x+1)+A_2(x+1)+A_3x^2$$

Taking $x=0$ we obtain

$$-1 = A_2.$$

Taking $x=1$ we obtain

$$-2 = A_3.$$

Thus,

$$x-1 = A_1x(x+1)-(x+1)-2x^2.$$

Taking $x=1$ we obtain

$$A_1 = 2.$$

Hence,

$$\int \frac{x-1}{x^3+x^2}\,dx = 2\int\frac{dx}{x} - \int\frac{dx}{x^2} - 2\int\frac{dx}{x+1} = 2\ln|x| + \frac{1}{x} - 2\ln|x+1| + C.$$

(vii) $\displaystyle\int \frac{1}{(x+4)(x-1)}\,dx.$

Solution: Special Case, Partial Fractions: Since

$$\frac{1}{(x+4)(x-1)} = \frac{A_1}{x+4} + \frac{A_2}{x-1} = \frac{A_1(x-1)+A_2(x+4)}{(x+4)(x-1)}$$

we have that

$$1 = A_1(x-1)+A_2(x+4).$$

Taking $x=-4$ we get

$$A_1 = -\frac{1}{5}.$$

Taking $x=1$ we get

$$A_2 = \frac{1}{5}.$$

Thus

$$\int \frac{1}{(x+4)(x-1)}\,dx = -\frac{1}{5}\int\frac{dx}{x+4} + \frac{1}{5}\int\frac{dx}{x-1} = -\frac{1}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C.$$

e) Evaluate

(i) $\displaystyle\int_0^4 x^2 + 3x^{7/2}\,dx.$

Solution:

$$\int_0^4 x^2 + 3x^{7/2}\,dx = \left(\frac{x^3}{3} + \frac{2}{3}x^{9/2}\right)\Bigg|_0^4 = \frac{4^3 + 24^{9/2}}{3}.$$

(ii) $\displaystyle\int_1^2 \frac{(x+5)^2}{x^4}\,dx.$

Solution:

$$\int_1^2 \frac{(x+5)^2}{x^4}\,dx = \int_1^2 \frac{x^2+10x+25}{x^4}\,dx = \int_1^2 \frac{dx}{x^2} + 10\int_1^2 \frac{dx}{x^3} + 25\int_1^2 \frac{dx}{x^4} = \left(-\frac{1}{x} - \frac{10}{2x^2} - \frac{25}{3x^3}\right)\Bigg|_1^2 = 277/24.$$

f) Consider the definite integral $\displaystyle\int_1^5 e^{x^2}\,dx.$

Estimate the integral using midpoints and trapezoidal rule with four intervals. Draw a picture.

Solution: We need four subintervals, so

$$\Delta_x = \frac{5-1}{4} = 1.$$

Thus, our endpoints in the subintervals are

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 4, \quad x_4 = 5.$$

Midpoint:

$$\int_1^5 e^{x^2}\,dx \approx \Delta_x\left(f(3/2) + f(5/2) + f(7/2) + f(9/2)\right) = e^{9/4} + e^{25/4}e^{49/5} + e^{81/4}.$$

Trapezoidal:

$$\int_1^5 e^{x^2}\,dx \approx \frac{\Delta_x}{2}\left(f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right) = \frac{1}{2}\left(e + 2e^4 + 2e^9 + 2e^{16} + e^{25}\right).$$