MAT 1320D Calculus I
Midterm 2 Solution

Professor: Wanshun Wong

1. [2 points] Find the derivative of $f(x) = (\cos x)^x$.

We use logarithmic differentiation: $\ln f(x) = \ln(\cos x)^x = x \ln \cos x$. Differentiate both sides with respect to x, we get

$$\frac{1}{f(x)} \cdot f'(x) = \ln \cos x + x \cdot \frac{1}{\cos x} \cdot (-\sin x).$$

Therefore $f'(x) = (\cos x)^x(\ln \cos x - x \tan x)$.

2. [2 points] A spotlight on the ground shines on a wall 6 m away. A boy of height 1 m is walking from the spotlight toward the wall. How fast the length of his shadow on the wall is changing when he is 4 m away from the light and walking at a speed 0.5 m/s?

Let x be the distance between the spotlight and the boy, and y be the length of the shadow. By comparing similar triangles we get

$$\frac{y}{6} = \frac{1}{x},$$

hence $y = \frac{6}{x}$, and $y' = -\frac{6}{x^2} \cdot x'$. When $x = 4$, we get $y' = -\frac{6}{16} \cdot \frac{1}{2} = -\frac{3}{16}$.

3. [1 point] What is $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^2} \frac{t^2 + e^t}{\sqrt{t^3 + \tan t}}\, dt\right)$?

This is an application of Fundamental Theorem of Calculus and Chain Rule, the answer is $\dfrac{x^4 + e^{(x^2)}}{\sqrt{x^6 + \tan x^2}} \cdot 2x$.

4. [5 points] Evaluate the following integrals:

(a) $\displaystyle\int \arctan(2x)\, dx$

First we use substitution $y = 2x, dy = 2\, dx$, and

$$\int \arctan(2x)\, dx = \frac{1}{2} \int \arctan y\, dy.$$

Then we use integration by parts with $u = \arctan y$, $du = \frac{1}{1+y^2}\,dy$, $v = y$, $dv = dy$.

$$\frac{1}{2}\int \arctan y\,dy = \frac{1}{2}\cdot\left(y\arctan y - \int \frac{y}{1+y^2}\,dy\right)$$

where the last integral can be computed by substitution easily. Therefore the final answer is

$$x\arctan(2x) - \frac{1}{4}\ln(4x^2+1) + C.$$

(b) $\displaystyle\int x^{10}\ln x\,dx$

We use integration by parts with $u = \ln x$, $du = \frac{1}{x}\,dx$, $v = \frac{x^{11}}{11}$, $dv = x^{10}\,dx$. So

$$\int x^{10}\ln x\,dx = \frac{x^{11}\ln x}{11} - \int \frac{x^{10}}{11}\,dx,$$

and the final answer is $\displaystyle\frac{x^{11}\ln x}{11} - \frac{x^{11}}{121} + C$.

(c) $\displaystyle\int \frac{1}{(x^2-2x+2)^{3/2}}\,dx$

By completing square,

$$\int \frac{1}{(x^2-2x+2)^{3/2}}\,dx = \int \frac{1}{((x-1)^2+1)^{3/2}}\,dx.$$

Then we use substitution $y = x-1$ to get

$$\int \frac{1}{((x-1)^2+1)^{3/2}}\,dx = \int \frac{1}{(y^2+1)^{3/2}}\,dy.$$

The last step is to use trigonometric substituation $y = \tan\theta$, $dy = \sec^2\theta\,d\theta$.

$$\begin{aligned}\int \frac{1}{(y^2+1)^{3/2}}\,dy &= \int \frac{1}{(\sec^2\theta)^{3/2}}\sec^2\theta\,d\theta \\ &= \int \cos\theta\,d\theta \\ &= \sin\theta + C.\end{aligned}$$

Rewriting the answer in terms of x (by drawing a right angled triangle with opposite side and adjacent side of length $y = x - 1$ and 1 respectively), the final answer is $\dfrac{x-1}{\sqrt{x^2-2x+2}} + C$.

(d) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx$

Let $u = \cos x$, $du = -\sin x \, dx$. When $x = 0, u = 1$; when $x = \pi/2, u = 0$.

$$\begin{aligned}\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx &= \int_0^{\pi/2} \sin x (1 - \cos^2 x)(\cos^4 x) \, dx \\ &= \int_1^0 -(1-u^2)(u^4) \, du \\ &= \frac{u^5}{5} - \frac{u^7}{7}\Big|_0^1 = \frac{2}{35}\end{aligned}$$

(e) $\displaystyle\int_{\pi^2/4}^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx$

Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} \, dx$. When $x = \pi^2/4, u = \pi/2$; when $x = \pi^2, u = \pi$.

$$\begin{aligned}\int_{\pi^2/4}^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx &= \int_{\pi/2}^{\pi} 2\cos u \, du \\ &= 2\sin u|_{\pi/2}^{\pi} = -2.\end{aligned}$$

5. [2 points] Find the linear approximation of $f(x) = \sqrt{9+x}$ at $x = 0$ and use it to estimate $\sqrt{9.5}$.

We have $f'(x) = \dfrac{1}{2\sqrt{9+x}}$, so $f'(0) = \frac{1}{6}$. The linear approximation is

$$f(x) \approx 3 + \frac{1}{6}(x - 0) = 3 + \frac{x}{6}.$$

Then $\sqrt{9.5} = f(0.5) \approx 3 + \frac{0.5}{6} = \frac{37}{12}$.

6. [2 points] Use R_4 to estimate the area under the curve $y = \dfrac{1}{1+4x}$ between $x = 0$ and $x = 1$.

We have $\Delta x = \frac{1}{4}$, and the right endpoints of the subintervals are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1. Therefore

$$R_4 = \frac{1}{4}(\frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{1+3} + \frac{1}{1+4}) = \frac{77}{240}.$$

7. [1 point] What is the area under the curve $y = \dfrac{1}{1+4x}$ between $x = 0$ and $x = 1$?

We have

$$A = \int_0^1 \frac{1}{1+4x}\,dx = \frac{\ln(1+4x)}{4}|_0^1 = \frac{\ln 5}{4}.$$