University of Ottawa

Department of Mathematics and Statistics

MAT 1302D : Mathematical Methods II

Professor: Eric Hua

First Midterm Exam – Version A

Jan 31, 2012

Surname ______________________ First Name ______________________

Student # ______________________

DGD # (1=VNR 5070; 2=LMX 124; 3=TBT 070; 4=DMS 1110) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(f) You are strongly recommended to write in **pen**, not pencil.
(g) You have to show your work for each question.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	5	5	5	5	5	5	30
Grade							

1. Consider the following linear system

$$\begin{cases} x_1 + 2x_2 - x_3 + 5x_4 = 1 \\ 2x_1 + 2x_2 - 2x_3 + 6x_4 = 3 \\ x_1 + x_2 - x_3 + 3x_4 = 2 \end{cases}$$

(a) [**4 points**] Row reduce the augmented matrix to the reduced row echelon form.

(b) [**1 point**] Determine if the linear system is consistent or inconsistent. (If the system is consistent, you do not need to find the general solution.)

Solution: (a)

We write the augmented matrix of the system and row reduce it

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 2 & 2 & -2 & 6 & 3 \\ 1 & 1 & -1 & 3 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & -1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0.5 \end{array}\right] \xrightarrow{R_3 \to 2R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 0 \\ 0 & -2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(b) Since the rightmost column is a pivot column, the linear system is inconsistent.

2. [**5 points**] Determine all values of h such that the vector $\begin{bmatrix} -1 \\ h \\ 6 \end{bmatrix}$ belongs to Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix} \right\}$.

Solution: We write the corresponding augmented matrix and then row reduce it:

$$\left[\begin{array}{cc|c} 1 & 3 & -1 \\ -1 & 1 & h \\ 0 & -8 & 6 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 4 & h-1 \\ 0 & -8 & 6 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 4 & h-1 \\ 0 & 0 & 2h+4 \end{array}\right]$$

The vector $\begin{bmatrix} -1 \\ h \\ 6 \end{bmatrix}$ belongs to Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix} \right\}$ if and only if the system with the latter matrix as the augmented matrix is consistent, that is, the rightmost column is not a pivot column. This is equivalent to $2h + 4 = 0$, or $h = -2$.

3. [**5 points**] Find the general solution to the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 & -1 & 1 \\ -2 & 4 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

You should write your solution in **vector parametric form**.

Solution: Assume $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. We row reduce the augmented matrix of the corresponding linear system.

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 1 \\ -2 & 4 & 3 & 1 & -4 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 4 & -1 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right]$$

x_1, x_3 basic. x_2, x_4 free. Therefore the general solution is:

$$\begin{cases} x_1 = 2x_2 - 4x_4 - 1 \\ x_2 = \text{free} \\ x_3 = -3x_4 - 2 \\ x_4 = \text{free} \end{cases}$$

and the vector parametric form of the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 - 1 \\ x_2 \\ -3x_4 - 2 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

4. [**5 points**] Consider the following linear system

$$\begin{cases} x_1 - x_2 &= 2 \\ 2x_1 + hx_2 &= k. \end{cases}$$

(i) For what values of h and k does the system have no solution?

(ii) For what values of h and k does the system have only one solution?

(iii) For what values of h and k does the system have infinitely many solution?

Solution:

$$[A|\vec{b}] = \begin{bmatrix} 1 & -1 & 2 \\ 2 & h & k \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & h+2 & k-4 \end{bmatrix}.$$

(i) $h = -2, k \neq 4$;

(ii) $h \neq -2, k \in \mathbb{R}$;

(iii) $h = -2, k = 4$;

5. [**5 points**] Consider the following system of linear equations

$$\begin{cases} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7kx_3 &= 0 \\ -3x_1 - 9x_2 + 11kx_3 &= 0 \end{cases}$$

Find all values of k such that the system has only the trivial solution.

Solution:

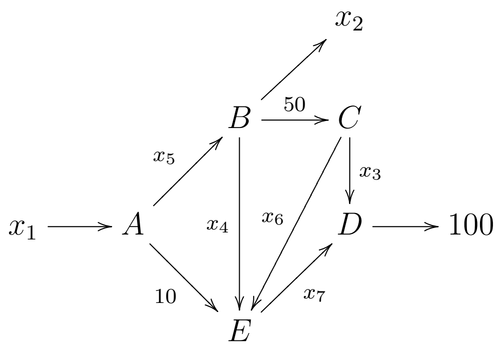
$$\text{The augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & 5 & -7k & 0 \\ -3 & -9 & 11k & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -3 & -7k+16 & 0 \\ 0 & 3 & 11k-24 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -3 & -7k+16 & 0 \\ 0 & 0 & 4k-8 & 0 \end{bmatrix}$$

Hence, for $4k - 8 \neq 0$, i.e., $k \neq 2$, the system has only trivial solution.

6. Consider the traffic flow described by the following diagram. The letters A through E label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.



(a) **[3 points]** Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 7$. (Do not solve the linear system at this stage.)

Solution:

A	x_1	$= x_5 + 10$
B	x_5	$= x_2 + x_4 + 50$
C	50	$= x_3 + x_6$
D	$x_3 + x_7$	$= 100$
E	$10 + x_4 + x_6$	$= x_7$
Total	x_1	$= x_2 + 100$

(b) [**1 point**] The reduced row echelon form of the augmented matrix corresponding to the linear system in part (a) is:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -90 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Which variables are free?

Solution: x_5, x_7.

(c) [**1 point**] Find the general solution of the linear system by filling in the following blank:

Solution: x_1, x_2, x_3, x_4, x_6: basic; x_5, x_7: free.

$$\begin{cases} x_1 = x_5 + 10 \\ x_2 = x_5 - 90 \\ x_3 = -x_7 + 100 \\ x_4 = 40 \\ x_5 = \text{free} \\ x_6 = x_7 - 50 \\ x_7 = \text{free} \end{cases}$$