NAME (underline family name):

STUDENT NUMBER: SIGNATURE:

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 263

ORDINARY DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

G Schmidt
S.W. Drury

Examiner: G. Schmidt **Date:** Tuesday, April 18, 2006
Associate Examiner: S. Drury **Time:** 9:00 AM - 12:00 PM

Instructions

1. Write your name and student number on this examination script.
2. No books, calculators or notes allowed.
3. Regular and or Translation dictionary is permitted.
4. This examination booklet consists of this cover page, 9 pages of questions and 2 blank pages (11 numbered pages in all). Please take a couple of minutes in the beginning of the examination to scan the problems. (Please inform the invigilator if the booklet is defective.)
5. Answer all questions. You are expected to show all your work. All solutions are to be written on the page where the question is printed. You may continue your solutions on the facing page. If that space is exhausted you may continue on the blank pages at the end, clearly indicating any continuation on the page where the question is printed.
6. Your answers may contain expressions that cannot be computed without a calculator.
7. Circle your answers where confusion could arise.

GOOD LUCK!

Score Table

Problem	Points	Out of
1.		10
2.		10
3.		10
4.		10
5.		12
6.		12
7.		14
8.		14
9.		8
Total:		100

Solutions

1. (10 marks) Find the solution $y(x)$ of

$$xy' + y = \ln x, \ y(1) = 3.$$

Note $xy' + y = (xy)'$ so equation becomes

$(xy)' = \ln x$.

Hence $xy = \int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx$

(by integration by parts) $= x \ln x - x + C$

$$y(x) = \ln x - 1 + \frac{C}{x}$$

$$3 = y(1) = \cancel{\ln 1} - 1 + C \quad \text{so } C = 4$$

$$\boxed{y(x) = \ln x - 1 + \frac{4}{x}}$$

(If you do not notice 1st step rewrite equation as $y' + \frac{1}{x} y = \frac{1}{x} \ln x$.

Multiply by $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ to get back to original equation!).

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2. (10 marks) Using the substitution $v = y/x$, find the solution $y(x)$ of

$$xyy' = x^2 + y^2,\ y(1) = 2.$$

Equation can be rewritten as

$$y' = \frac{x}{y} + \frac{y}{x} = \frac{1}{v} + v$$

$$v' = \frac{y'}{x} - \frac{1}{x^2}y = \frac{1}{x}\left[y' - v\right] = \frac{1}{vx} \text{ (separable)}$$

$$\int v\,dv = \int \frac{1}{x}\,dx \quad \text{so } \frac{1}{2}v^2 = \ln|x| + c$$

$$v^2 = 2\ln|x| + 2c \quad \text{or } y^2 = 2x^2(\ln|x| + C)$$

$$y(1) = 2 \text{ gives } 4 = 2C \quad \text{so } C = 2$$

$y^2 = 2x^2(\ln|x| + 2)$. Since $y(1) > 0$ take positive square ~~root~~

$$\boxed{y(x) = \sqrt{2x^2(\ln|x| + 2)} = \sqrt{2}\, x \sqrt{\ln x + 2}}$$

for $x > 0$

4. (10 marks) Find the solution $y(x)$ of

$$y'' + y = x\cos x - \cos x,\ y(0) = 2,\ y'(0) = 1.$$

$y_h = C_1 \cos x + C_2 \sin x$ (since $r^2+1=0$ has roots $r = \pm i$)

Try $y_p(x) = x(A+Bx)\cos x + x(C+Dx)\sin x$

$= (Ax+Bx^2)\cos x + (Cx+Dx^2)\sin x$

$y_p'(x) = [A + 2Bx + Cx + Dx^2]\cos x$
$+ [-Ax - Bx^2 + C + 2Dx]\sin x$

$y_p''(x) = [2B + C + 2Dx - Ax - Bx^2 + C + 2Dx]\cos x$
$+ [-A - 2Bx - Cx - Dx^2 - A - 2Bx + 2D]\sin x$

$y_p''(x) + y_p = [2B + 2C + 4Dx]\cancel{2Ax}\cos x$
$+ [-2A + 2D - 4Bx]\sin x$

$= x\cos x - \cos x$

if $A = \cancel{D}$, $B = 0$, $C = -\frac{1}{2}$, $D = \frac{1}{4}$, $A = \frac{1}{4}$

So $y_p = \frac{1}{4}x\cos x + (-\frac{1}{2}x + \frac{1}{4}x^2)\sin x$

So $y = C_1\cos x + C_2\sin x + \frac{1}{4}x\cos x + (-\frac{1}{2}x + \frac{1}{4}x^2)\sin x$

$y' = -C_1\sin x + C_2\cos x + \frac{1}{4}\cos x - \frac{1}{4}x\sin x$
$(-\frac{1}{2} + \frac{1}{2}x)\sin x + (-\frac{1}{2}x + \frac{1}{4}x^2)\cos x$

$y(0) = C_1 = 2$

$y'(0) = C_2 + \frac{1}{4} = 1$, So $C_2 = \frac{3}{4}$

$$\boxed{y(x) = 2\cos x + \frac{3}{4}\sin x + \frac{1}{4}x\cos x + (-\frac{1}{2}x + \frac{1}{4}x^2)\sin x}$$

5. (12 marks) Find the general solution $y(x)$ of

$$y'' + 3y' + 2y = \frac{1}{1+e^x}.$$

characteristic equation

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0 \qquad r = -2 \text{ or } r = -1$$

$$y_h = Ae^{-2x} + Be^{-x}$$

$$W(x) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$W_{(1)}(x) = \begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix} = -\frac{e^{-2x}}{1+e^x}$$

$$\frac{W_{(1)}(x)}{W(x)} = \frac{e^x}{1+e^x}$$

$$\frac{W_{(2)}(x)}{W(x)} = \frac{1}{1+e^x}\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix} = -\frac{e^{2x}}{1+e^x} = \frac{e^x}{1+e^x} - e^x$$

$$y_p = e^{-x}\int \frac{W_{(1)}(x)}{W(x)}dx + e^{-2x}\int \frac{W_{(2)}(x)}{W(x)}dx$$

$$= e^{-x}\ln(1+e^x) + e^{-2x}\left[\ln(1+e^x) - e^x\right]$$

$$\boxed{y = y_p + y_h = e^{-x}\ln(1+e^x) + e^{-2x}\left[\ln(1+e^x) - e^x\right] + Ae^{-x} + Be^{-2x}}$$

6. (12 marks) Use Laplace transforms, and the table which follows, to solve

$$y'' + y = f(t),\ y(0) = 0, y'(0) = 2,$$

where $f(t) = 2$ for $0 \le t \le \pi$ and 1 for $t > \pi$.

function $f(t)$	Laplace transform $F(s)$
1	$1/s \ (s > 0)$
t^n	$n!/s^{n+1} \ (s > 0)$
e^{at}	$1/(s-a) \ (s > a)$
$\sin at$	$a/(s^2+a^2) \ (s > 0)$
$\cos at$	$s/(s^2+a^2) \ (s > 0)$
$e^{-at}f(t)$	$F(s+a)$
$H(t-a)$ or $u_a(t)$ $(a \ge 0)$	$e^{-as}/s \ (s > 0)$
$\delta(t-a)$ $(a > 0)$	e^{-as}
$H(t-a)f(t-a)$ or $u_a(t)f(t-a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - f^{(n-1)}(0)$
$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$

$$f(t) = 2(1 - H(t-\pi)) + H(t-\pi)$$
$$= 2 - H(t-\pi)$$

$$\mathcal{L}(f)(s) = \frac{2}{s} - \frac{e^{-\pi s}}{s}$$

$$\begin{cases} y'' + y = f(t) \\ y(0) = 0 \quad y'(0) = 2 \end{cases}$$ implies

$$s^2 Y - 2 + Y = \frac{2}{s} - \frac{e^{-\pi s}}{s}$$

$$Y = \frac{2}{s(s^2+1)} + \frac{2}{s^2+1} + \frac{e^{-\pi s}}{s(s^2+1)}$$

$$= 2\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{2}{s^2+1} + e^{-\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$$

$$y(t) = 2 - 2\cos(t) + 2\sin(t) - H(t-\pi) + \cos(t-\pi)H(t-\pi)$$

$$\boxed{y(t) = 2\left[1 + \sin(t) - \cos(t)\right] + H(t-\pi)\left[\cos(t-\pi) - 1\right]}$$

or

$$\boxed{y(t) = 2\left[1 + \sin(t) - \cos(t)\right] + H(t-\pi)\left[1 + \cos(t)\right]}$$

7. (14 marks in total) Let

$$A = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.$$

(a) (4 marks) Given that 3 is an eigenvalue find a basis for the corresponding eigenspace.
(b) (4 marks) Find the other eigenvalue(s) and bases for their eigenspace(s).
(c) (4 marks) Find orthogonal Q and diagonal D such that $A = QDQ^T$.
(d) (2 marks) Give a geometric interpretation of the transformation from $\mathbf{R}^3$ to itself defined by the matrix, $\frac{1}{3}A$.

(a) E_3? Solve $[A-3I]\vec{x} = \vec{0}$ or $\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

By inspection or otherwise find $E_3 = \text{Span}\{(1,-1,0),(0,1,-1)\}$

(b) Various alternatives:

- $\det[A-\lambda I] = -[\lambda^3 - 3\lambda^2 - 9\lambda + 27]$ (after some calculation)
  $= -(\lambda-3)^2(\lambda+3)$ (know 3 is a double root so -3 is 3rd root follows easily).
- Because A is symmetric 3rd eigenvector is
  $(1,-1,0) \times (0,1,-1) = (1,1,1)$. $A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-3\\-3\\-3\end{pmatrix} = -3\begin{pmatrix}1\\1\\1\end{pmatrix}$
  So -3 is third eigenvalue.
- (If you know about trace, trace $A = 1+1+1 = \lambda_1+\lambda_2+\lambda_3 = 6+\lambda_3$
  so $\lambda_3 = -3$.

$E_{-3} = \text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ (Solve $(A-3I)\vec{x} = \vec{0}$ to get this.

(c) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. To get Q first find orthogonal basis for E_3: $(1,-1,0)$, $(0,1,-1) - \frac{(0,1,-1)\cdot(1,-1,0)}{2}(1,-1,0)$

$= (0,1,-1) + (\frac{1}{2},-\frac{1}{2},0) = (\frac{1}{2},\frac{1}{2},-1) = \frac{1}{2}(1,1,-2)$

$E_3 = \text{Span}\{(1,-1,0),(1,1,-2\}$. So, normalizing eigenvectors

$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ (not only possibility, depending on choices of eectors made for E_3).

(d) $\frac{A}{3}$ has eigenvalues $(1,1,-1)$ $E_1 = \text{Span}\{(1,-1,0),(0,1,-1)\}$
$E_{-1} = \text{Span}\{(1,1,1)\}$

$\frac{A}{3}$ is reflection through plane E_1.

8. (14 marks in total) Consider the matrix $A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$.

(a) (6 marks) Diagonalize A.

(b) (6 marks) Evaluate e^{At}, for t real.

(c) (2 marks) Using e^{At}, write down an expression for the solution of the system of the system of differential equations

$$\begin{aligned} x_1' &= 2x_2, \qquad & x_1(0) = a, \\ x_2' &= x_1 + x_2, \qquad & x_2(0) = b. \end{aligned}$$

(a) $\chi_A(\lambda) = \det \begin{bmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = \lambda(\lambda-1) - 2 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$

Eigenvalues are $2, -1$.

E_2 is solution space of $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $E_2 = \mathrm{Span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E_{-1} is solution space of $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $E_{-1} = \mathrm{Span}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

So $A = SDS^{-1}$ where $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

(b) $e^{At} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} & 2e^{2t} \\ e^{-t} & -e^{-t} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} e^{2t}+2e^{-t} & 2e^{2t}-2e^{-t} \\ e^{2t}-e^{-t} & 2e^{2t}+e^{-t} \end{pmatrix}$

(c) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{At}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{3}\begin{pmatrix} a(e^{2t}+2e^{-t}) + b(2e^{2t}-2e^{-t}) \\ a(e^{2t}-e^{-t}) + b(2e^{2t}+e^{-t}) \end{pmatrix}$

9. (8 marks in total) Let A be a real 3×3 matrix.
(a) (6 marks) Given that -2 and $3+2i$ are eigenvalues corresponding respectively to eigenvectors $(1,1,1)$ and $(1-3i, i, 2)$ find a basis of real solutions of the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.
(b) (2 marks) For which initial values $\mathbf{x}(0)$ does the corresponding solution $\mathbf{x}(t)$ satisfy $\mathbf{x}(t) \longrightarrow 0$ as $t \longrightarrow +\infty$?

(a) One real solution is $\bar{x}(t) = e^{-2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

A complex solution is

$$e^{(3+2i)t}\begin{pmatrix}1-3i\\ i\\ 2\end{pmatrix} = e^{3t}(\cos 2t + i\sin 2t)\begin{pmatrix}1-3i\\ i\\ 2\end{pmatrix}$$

$$= e^{3t}\begin{pmatrix}\cos 2t + 3\sin 2t + i[\sin 2t - 3\cos 2t]\\ -\sin 2t + i\cos 2t\\ 2\cos 2t + i\,2\sin 2t\end{pmatrix}$$

Taking real and imaginary parts gives

$$e^{3t}\begin{pmatrix}\cos 2t + 3\sin 2t\\ -\sin 2t\\ 2\cos 2t\end{pmatrix}, \quad e^{3t}\begin{pmatrix}\sin 2t - 3\cos 2t\\ \cos 2t\\ 2\sin 2t\end{pmatrix}$$

which together with $e^{-2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$ give a basis of real solutions.

(b) General solution is

$$x(t) = c_1 e^{-2t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}\cos 2t + 3\sin 2t\\ -\sin 2t\\ 2\cos 2t\end{pmatrix} + c_3 e^{3t}\begin{pmatrix}\sin 2t - 3\cos 2t\\ \cos 2t\\ 2\sin 2t\end{pmatrix}$$

$x(t) \to 0$ as $t \to \infty$ if $c_2 = c_3 = 0$. Then

$\bar{x}(0) = c_1\begin{pmatrix}1\\1\\1\end{pmatrix}$ so must have $\boxed{\bar{x}(0) \in \operatorname{Span}\begin{pmatrix}1\\1\\1\end{pmatrix}}$.