Solution:

Part 1: Multiple choice questions, 2 points each (no partial marks), chose the best possible answer.

[2] (1) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, (b) $\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$, (c) $\begin{bmatrix} 6 \\ 15 \end{bmatrix}$, (d) $\begin{bmatrix} 4 \\ 10 \\ 18 \end{bmatrix}$, (e) $\begin{bmatrix} 15 \\ 6 \end{bmatrix}$.

Solution: (c)

[2] (2) Given (i) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, (iii) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Which of them are in the span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$?

(a) (i) and (ii), (b) (i) and (iii), (c) (ii) and (iii), (d) (i), (ii) and (iii), (e) (iii) only.

Solution: (a)

[2] (3) Given

$$A = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

List all matrices which are in row echelon form but not in reduced row echelon form

(a) A, B, and D, (b) A and B, (c) A and D, (d) B and D,

(e) B only.

Solution: (e)

[2] (4) Given that $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution of the non-homogeneous system $A\vec{x} = \vec{b}$, and that $t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the general solution of the corresponding homogeneous system $A\vec{x} = \vec{0}$, where $t \in \mathbb{R}$, which of the following is a solution of $A\vec{x} = \vec{b}$?

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$; (b) $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$; (c) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$; (d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; (e) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$;

Solution: (e)

Part 2: Long Answer Questions (Steps are necessary!)

[3] (5) Consider the following linear system

$$\begin{aligned} x_1 - x_2 &= 2 \\ 2x_1 + hx_2 &= k. \end{aligned}$$

(i) For what values of h and k does the system have no solution?

(ii) For what values of h and k does the system have only one solution?

(iii) For what values of h and k does the system have infinite solution?

Solution:

$$[A|\vec{b}] = \begin{bmatrix} 1 & -1 & 2 \\ 2 & h & k \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & h+2 & k-4 \end{bmatrix}.$$

(i) $h = -2, k \neq 4$;

(ii) $h \neq -2, k \in \mathbb{R}$;
(iii) $h = -2, k = 4$;

[8] (6) Consider the following system of linear equations

$$\begin{aligned} x_1 - 8x_3 + 3x_4 &= 2 \\ x_2 - 7x_3 - 3x_4 &= 1 \end{aligned}$$

(i) [1 mark] What is the vector equation of the system?
(ii) [1 mark] What is the matrix equation of the system?
(iii) [2 marks] What is the augmented matrix of the system? Is it in reduced row echelon form?
(iv) [4 marks] Solve the system and write the general solution in parametric vector form.

Solution: (i) The vector equation of the system:

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -7 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(ii) The matrix equation of the system:

$$\begin{bmatrix} 1 & 0 & -8 & 3 \\ 0 & 1 & -7 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(iii)

$$\text{the augmented matrix} = \begin{bmatrix} 1 & 0 & -8 & 3 & 2 \\ 0 & 1 & -7 & -3 & 1 \end{bmatrix}.$$

It is in reduced row echelon form.
(iv) We have we have

$$\begin{aligned} x_1 &= 2 + 8x_3 - 3x_4 \\ x_2 &= 1 + 7x_3 + 3x_4 \\ x_3 &= free \\ x_4 &= free \end{aligned}$$

Let $x_3 = s$, $x_4 = t$, we have $x_1 = 2 + 8s - 3t$, $x_2 = 1 + 7s + 3t$. Thus

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 + 8s - 3t \\ 1 + 7s + 3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 8 \\ 7 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

[5] (7) Given four vectors $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -2 \\ -15 \\ 7 \end{bmatrix}$. Write $\vec{y}$ as a linear combination of $\vec{u}, \vec{v}, \vec{w}$.

Solution:

$$\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{y} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 & -2 \\ -1 & 8 & 5 & -15 \\ 1 & -2 & 5 & 7 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 3 & -1 \\ -1 & 8 & 5 & -15 \\ 1 & -2 & 5 & 7 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_1, R_3 - R_1} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 8 & 8 & -16 \\ 0 & -2 & 2 & 8 \end{bmatrix} \xrightarrow{\frac{1}{8}R_2} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & 2 & 8 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3, R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, $\vec{y}$ can be written as a linear combination of $\vec{u}, \vec{v}, \vec{w}$:

$$\vec{y} = -4\vec{u} - 3\vec{v} + \vec{w}.$$

[6] (8) Consider the following system of linear equations

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

(i) [3 marks] Reduce the augmented matrix to the reduced row echelon form.
(ii) [1 mark] Does the system have non-trivial solution? Explain.
(iii) [2 marks] Write the solution set in parametric vector form.

Solution:
(i)

$$\text{augmented matrix } = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & 5 & -7 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1, R_3 + 3R_1} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & -15 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 5 & -15 & 0 \end{bmatrix} \xrightarrow{R_3 - 5R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(ii) From (i) we have

$$\begin{aligned} x_1 + 4x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

Since it has free variable, it has non-trivial solutions.

(iii) We have

$$\begin{aligned} x_1 &= -4t \\ x_2 &= 3t \\ x_3 &= t(free) \end{aligned}$$

Then the general solution is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$