REVIEW

Chapter 1

§1.2 Separable DE

$$\boxed{g(y)\,y' = f(x)} \Rightarrow g(y)\frac{dy}{dx} = f(x) \Rightarrow \int g(y)\,dy = \int f(x)\,dx + \underset{\text{I.C.}}{C}$$

§1.3 DE with hom. coeff.

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$$

Recall: if $F(\lambda x, \lambda y) = \lambda^p F(x,y)$
$\Rightarrow$ F homog. of degree p

If M & N homog. of the same degree

$\Rightarrow$ substitution $\left\{\begin{array}{l} u = y/x \to \text{keep } u \,\&\, x \\ \underline{\text{or}} \\ u = x/y \to \text{keep } u \,\&\, y \end{array}\right\}$ will make DE separable.

§1.4 Exact DE

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$$

- If $\frac{\partial M}{\partial y} = M_y = N_x = \frac{\partial N}{\partial x} \Rightarrow$ DE is <u>exact</u>
- $\Rightarrow \exists\ F(x,y)$ s.t. $F_x = M$ & $F_y = N$

- To find $F \to F(x,y) = \int M\,dx + h(y)$ ($\underline{\text{or}}$ $F(x,y) = \int N\,dy + g(x)$)

$$\downarrow$$

$$F_y = \frac{\partial}{\partial y}\left[\int M\,dx\right] + h'(y) = N \to \text{Solve for } h(y)$$

Same idea for $g(x)$.

§1.5 Integrating Factor

$\boxed{M\,dx + N\,dy = 0}$

- If $M_y \neq N_x \Rightarrow$ Not Exact
- Then check: • $\dfrac{M_y - N_x}{N}$ fct of x only

  or

  • $\dfrac{M_y - N_x}{M}$ fct of y only

- Then int. fact. $\left.\begin{array}{l} \bullet\ \mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \\ \quad \text{or} \\ \bullet\ \mu(y) = e^{-\int \frac{M_y - N_x}{M} dy} \end{array}\right\} \rightarrow \begin{array}{l} M^* = \mu M \\ N^* = \mu N \end{array}$

- Now $M^* dx + N^* dy = 0$ is <u>exact</u>.

§1.6 First-Order Linear DE

- $\boxed{y' + f(x)y = r(x)}$ • Let $\mu(x) = e^{\int f(x)dx}$

  • G.S. $y(x) = \dfrac{\int \mu(x) r(x) dx + C}{\mu(x)}$

- <u>Bernoulli Eq</u> : $\boxed{y' + p(x)y = g(x)y^a}$

  • Let $u(x) = (y(x))^{1-a}$

  DE becomes $u' + (1-a)p(x)u = (1-a)g(x)$

  Linear DE → solve → come back to $y(x)$.

Chapter 2

Recall: • n^{th} order linear diff operator $L := a_n(x)D^n + \dots + a_1(x)D + a_0(x)$

$$Ly = a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y$$

• n^{th} order homog. linear DE $Ly = 0$ will have n l. ind sol that form the general sol. (G.S) $y(x) = C_1 y_1(x) + \dots + C_n y_n(x)$

Reduction of Order

$(*)$ $\boxed{y'' + p(x)y' + q(x)y = 0}$

• Suppose $y_1(x)$ is a sol to $(*)$

• we want $y_2(x)$ sol to $*$ but $\dfrac{y_2(x)}{y_1(x)} = u(x) \neq cst$

• ultimately ... $y_2(x) = u(x)y_1(x) = y_1(x) \displaystyle\int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx$

G.S. $y(x) = C_1 y_1 + C_2 y_2$.

"See Chap. 3 for the rest"

Chapter 3

OverView Chap. 2–4

Chapter 2 Second order Homog. Linear DE.

-Special Case of Chap. 3-

Chapter 3 Linear DE of arbitrary order

$$Ly = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = r(x)$$

- Homog. case ($r(x) \equiv 0$) cst. coeff. ($a_i(x) \equiv a_i \in \mathbb{R}$)
  - → Find char eq. $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$
  - → Find the roots $\lambda_1, \dots, \lambda_n$
  - → Write the general sol. taking care of root multiplicity and complex conjugateness. $y(x) = c_1 y_1 + \dots + c_n y_n$.

- Nonhomog Case ($r(x) \not\equiv 0$)
  - → Find $y_h(x)$ → G.S. of the homog. case. $Ly = 0$
  - → G.S. is $y_g = y_h + y_p$ where y_p is a particular sol to $Ly = r(x)$ to $Ly = r(x)$

To Find y_p

→ Undetermined Coeff.

- if Ly has cst. coeff
- if $r(x)$ has finite # of l. ind deriv.

→ Var. of Param.

- always works but messy

- Euler-Cauchy Eq.

$$Ly = x^n y^{(n)} + x^{n-1} y^{(n-1)} + \dots + xy' + y$$

(We deal with $n=2$, $n=3$)

hapter 4 — Systems of diff eq.

- Reduction of a higher order lin. DE.

$$Ly = y^{n} + a_{n-1}(x)\,y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = r(x)$$

$$u = \begin{bmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{bmatrix}, \quad \begin{array}{l} u_1 = y \\ u_2 = y' \\ \vdots \\ u_n = y^{(n-1)} \end{array} \Rightarrow u' = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ -a_0(x) & \cdots & & -a_{n-1}(x) \end{bmatrix}}_{A(x)} u + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ r(x) \end{bmatrix}}_{g(x)}$$

- Homog cst. matrix syst. n=2

- A 2x2 cst. matrix, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.
- $u' = Au \rightarrow$ find eigenvals & eigenvect. λ_1, v_1; λ_2, v_2.

$\rightarrow$ G.S. $u(x) = C_1 e^{\lambda_1 x} v_1 + C_2 e^{\lambda_2 x} v_2$.

Beware of special cases for eigenvals.

- Nonhomog cst. coeff. syst. n=2 (Undeter. Coeff)

- $g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$ with finite # of deriv.
- $u' = Au + g(x)$
- Apply undetermined coeff as for scalar eq. but replace csts. by vectors in the construction of u_p.

Chapter 6 : Laplace Transforms

1 Definition

The Laplace Transform of a transformable function $f(t)$, $\mathcal{L}(f)$ is define the following way :

$$\mathcal{L}(f(t))(s) = \int_0^\infty e^{-st} f(t) dt \quad s > \gamma.$$

$\mathcal{L}$ is *linear* (i.e.: $\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha\mathcal{L}(f(t)) + \beta\mathcal{L}(g(t))$) and has an *inverse transform* ($\mathcal{L}^{-1}$).

2 Building Blocks

Here are some transforms of common functions :

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
t^n	$n!/s^{n+1}$	; $n = 0, 1, 2, \ldots$ and $s > 0$
e^{at}	$1/(s-a)$	; $s > a$
$\sin(kt)$	$k/(s^2+k^2)$	; $s > 0$
$\cos(kt)$	$s/(s^2+k^2)$	; $s > 0$
$\sinh(kt)$	$k/(s^2-k^2)$	; $s > k$
$\cosh(kt)$	$s/(s^2-k^2)$	; $s > k$
$\delta(t-a)$	e^{-as}	; $s > 0$
$u(t-a)$	$\frac{e^{-as}}{s}$	; $s > 0$

Here, $\delta(t-a)$ is the *Delta-Dirac* function shifted at a and $u(t-a)$ is the *Heaviside* function shifted at a defined as follows :

$$\delta(t-a) = \begin{cases} 0 & \text{if } t \neq a \\ \infty & \text{if } t = a \end{cases}$$

$$u(t-a) = \begin{cases} 0 & \text{if } t \leq a \\ 1 & \text{if } t > a \end{cases}$$

3 Transforms of Derivatives and Integrals

$$\mathcal{L}[f^{(n)}(t)](s) = s^n \mathcal{L}(f(t))(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

$$\mathcal{L}[\int_0^t f(\tau)d\tau](s) = \frac{1}{s}\mathcal{L}(f(t))(s)$$

4 Shifting Theorems

Suppose for the following that $F(s) = \mathcal{L}(f(t))$.

Theorem 1 (First Shifting Theorem)

$$\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t) \textit{ or } \mathcal{L}(e^{at}f(t)) = F(s-a).$$

Theorem 2 (Second Shifting Theorem)

$$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s) \textit{ or } \mathcal{L}^{-1}[e^{-as}F(s)] = u(t-a)f(t-a)$$

5 Derivatives and Integrals of Transforms

Theorem 3

$$F^{(n)}(s) = (-1)^n \mathcal{L}(t^n f(t))(s) \textit{ or } \mathcal{L}^{-1}(F^{(n)}(s)) = (-1)^n t^n f(t)$$

Theorem 4 *If* $\lim_{t\to 0^+} \frac{f(t)}{t}$ *exists, then*

$$\mathcal{L}[\frac{f(t)}{t}] = \int_s^\infty F(\xi)d\xi \textit{ or } \mathcal{L}^{-1}[\int_s^\infty F(\xi)d\xi] = \frac{1}{t}f(t)$$

6 Convolution

The *Convolution* of a function $f(t)$ with a function $g(t)$ is :

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

The *Convolution* is *distributive*, *associative* and most importantly *comutative*.

Theorem 5

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \ or\ \mathcal{L}^{-1}(F(s)G(s)) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}(G(s))$$

7 Laplace Transforms and Differential Equations

The use of the *Laplace Transform* to solve a differential equation can be described as follows:

1.Let $Y(s) = \mathcal{L}(y(t))$ where $y(t)$ is the (yet unknown) solution to the differential equation.

2.Find an expression for $Y(s)$.

3.Compute $\mathcal{L}^{-1}(Y(s))$ to come back to $y(t)$.

Note that these steps often make use of the results presented earlier in this document.