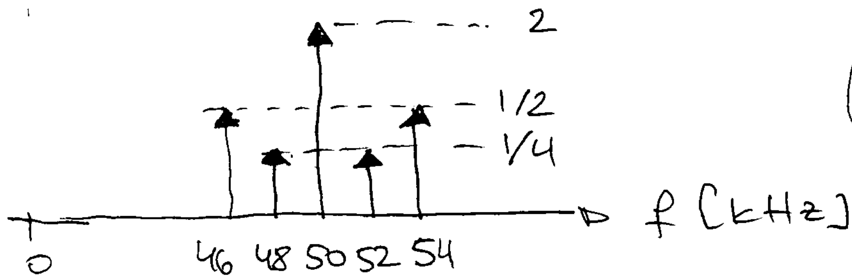
ELE 635 Communication Systems – Spring/Summer 2

$\boxed{\text{SOLUTIONS for MIDTERM EXAMINATION}}$

① Given that $\varphi_{AM}(t) = [A_c + m(t)]\cos\omega_c t$ with $\Phi_{AM}(f)$:



(For $f \geq 0$ only!)

(a) Since $\Phi_{AM}(f)$, $f \geq 0$ is symmetric with respect to $f = f_c$ we recognize that the component at $f = 50$ kHz is the carrier, and $m(t)$ is a two-tone signal. Also using the Fourier transform pair:

$$\cos\omega_0 t \longleftrightarrow \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

we identify the following parameters and signals:

- $\boxed{f_c = 50 \text{ kHz}}$
- $\boxed{A_c = 2\,(2) = 4}$
- $m(t) = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$ with

$$\left.\begin{aligned} A_1 &= 4(1/4) = 1, & f_1 &= 2 \text{ kHz} \\ A_2 &= 4\cdot(1/2) = 2, & f_2 &= 4 \text{ kHz}. \end{aligned}\right\}$$

Note: The factor "4" is due to the modulation, i.e. $A_1 \cos\omega_1 t \cos\omega_c t$ will result in $\frac{A_1}{2}\cos(\omega_c+\omega_1)t + \frac{A_1}{2}\cos(\omega_c-\omega_1)t$

i.e., $\boxed{m(t) = \cos 2\pi(2000)t + 2\cos 2\pi(4000)t}$

b) Let $E(t) = A_c + m(t)$ such that

$$E_{MAX} = \max E(t) = A_c + 1 + 2 = A_c + 3$$

$$E_{MIN} = \min E(t) = A_c - 1 - 2 = A_c - 3$$

It then follows:

$$\mu = \frac{E_{MAX} - E_{MIN}}{2A_c} = \frac{(A_c + 3) - (A_c - 3)}{2A_c} = \frac{6}{8} = \frac{3}{4}$$

$$\mu_+ = \frac{E_{MAX} - A_c}{A_c} = \frac{(A_c + 3) - A_c}{A_c} = \frac{3}{4}$$

$$\mu_- = \frac{A_c - E_{min}}{A_c} = \frac{A_c - (A_c - 3)}{A_c} = \frac{3}{4}$$

$$\therefore \boxed{\mu = \mu_+ = \mu_- = 3/4}$$

This result is expected as $m(t)$ is symmetric with respect to the time-axis.

(c) $R_L = 1\text{-}\Omega$ (normalized load).

Carrier: $A_c \cos w_c t \implies P_c = \frac{A_c^2}{2R_L} = \frac{4^2}{2} = 8$ W

Sidebands: $m(t) \cos w_c t \implies P_s = \frac{P_m}{2}$

with $P_m = \frac{1}{R_L}\left[\frac{A_1^2}{2} + \frac{A_2^2}{2}\right] = \frac{(1)^2}{2} + \frac{(2)^2}{2} = \frac{1}{2} + 2 = \frac{5}{2}$ W

Hence

$$P_s = \frac{1}{2}\left(\frac{5}{2}\right) = 5/4 \text{ W}$$

Total power:

$$P_{\varphi AM} = P_s + P_c = 5/4 + 8 = \frac{5 + 8(4)}{4} = \frac{37}{4} \text{ W}$$

Power Efficiency:

$$\eta = \frac{P_s}{P_{\varphi AM}} = \frac{5/4}{37/4} = 5/37$$

$$\therefore \boxed{\begin{array}{ll} P_s = 5/4 \text{ W into } 1\text{-}\Omega & P_{\varphi AM} = 37/4 \text{ W into } 1\text{-}\Omega \\ P_c = 8 \text{ W into } 1\text{-}\Omega & \eta = 5/37 \end{array}}$$

We can improve η by increasing to modulation index. Assuming that we do not want to overmodulate, i.e., $\mu > 1$, we can increase the amplitude of the modulating signal $m(t)$ to increase μ up to 1 (observe that in this problem $m(t)$ is symmetric with respect to horizontal axis hence $\mu = \mu_+ = \mu_-$ as we determined in part (a), hence it does not make any difference which "μ" we are referring to). To summarize replace $m(t)$ with $Km(t)$, $K \in \mathbb{R}^+$, such that the resulting modulation index $\mu = 1$.

The resulting η will still be less than 33%, which can only be achieved in the case of single-tone modulation at 100% modulation.

(d) Envelope detector requires $\mu \leq 1$. Hence, if $m(t)$ is replaced by $Km(t)$ with $K \in \mathbb{R}^+$, the maximum value of K is obtained when $\mu = 1$.

Let $m'(t) = Km(t)$, then

$$E'_{MAX} = \max_t \left[A_c + m'(t)\right] = A_c + K \max_t m(t)$$

$$= A_c + K(1+2) = 4 + 3K$$

$$E'_{min} = \min_t \left[A_c + m'(t)\right] = 4 - 3K$$

$$\mu' = \frac{E'_{MAX} - E'_{MAX}}{2A_c} = \frac{6K}{2(4)} = \frac{3}{4}K$$

We want $\mu' = 1 \Rightarrow \frac{3}{4}K = 1 \Rightarrow \boxed{K_{MAX} = 4/3}$

Power calculations (into $R_L = 1\text{-}\Omega$) resulting from $m'(t)$ will be as follows:

$$P_c = \text{unchanged} = 8W$$

$$P_S = \frac{P_{m'}}{2} = \frac{1}{2}K^2 P_M = K^2 \, 5/4 \; W$$

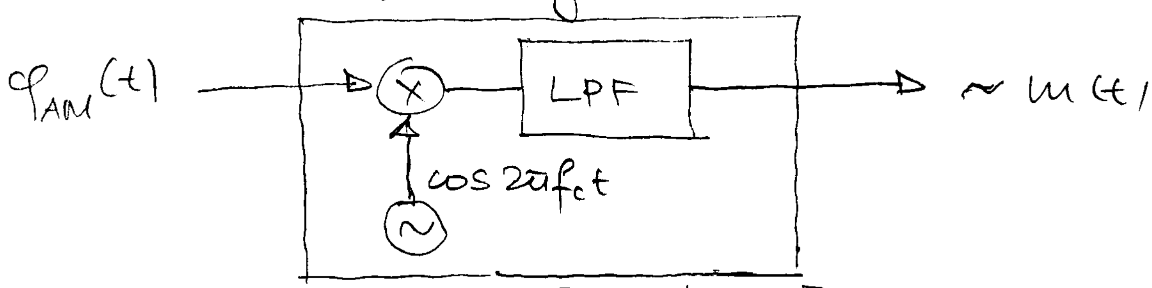
such that

$$\eta' = \frac{P_s}{P_c + P_s} = \frac{k^2\,5/4}{8 + k^2\,5/4} = \frac{5k^2}{32 + 5k^2}\Bigg|_{k=4/3} = \frac{(5)(16)}{(32)(9)+(5)(16)}$$

$$\boxed{\eta' = \frac{5}{23}}$$

Observe that $\eta = 5/37 \approx 14\%$ and $\eta' \approx 22\%$ a significant improvement in the power efficiency of the AM signal.

(e) If $m(t) \longrightarrow 2k\,m(t)$ where k was the value that would result in $\mu = 1$, then $2k\,m(t)$ will generate an overmodulated AM signal. We have to use a coherent detector to demodulate an overmodulated AM signal:



with LPF: passband $[0, 4^{+}\,\text{kHz}]$ (slightly greater than 4 kHz).

(f) The AM signal consists of the following sinusoidal components

$$\varphi_{AM}(t) = A_c \cos w_c t + (\cos w_1 t + 2\cos w_2 t)\cos w_c t \quad \text{with } \begin{array}{l} f_1 = 2\,\text{kHz} \\ f_2 = 4\,\text{kHz} \end{array}$$

or equivalently

$$\begin{aligned}\varphi_{AM}(t) = A_c \cos w_c t & + \frac{1}{2}\cos(w_c + w_1)t + \frac{1}{2}\cos(w_c - w_1)t \\ & + \cos(w_c + w_2)t + \cos(w_c - w_2)t\end{aligned}$$

Now, it is easy to see the effects of transmitting $\varphi_{AM}(t)$ over $H(f)$:

Frequency [rad]	[kHz]	Channel Gain $\lvert H(f)\rvert$	Phase $\arg[H(f)]$	Time-Delay slope of $\arg[H(f)]$
ω_c	50	G	$-2\pi(50000)t_1$	t_1
$\omega_c+\omega_1$	52	G	$-2\pi(52000)t_1$	t_1
$\omega_c-\omega_1$	48	G	$-2\pi(48000)t_1$	t_1
$\omega_c+\omega_2$	54	G	$-2\pi(54000)t_0$	t_0
$\omega_c-\omega_2$	46	G	$-2\pi(46000)t_0$	t_0

Hence, the output of $H(f)$ becomes

$$y(t) = GA_c\cos\omega_c(t-t_1)$$
$$+\frac{G}{2}\cos(\omega_c+\omega_1)(t-t_1) + \frac{G}{2}\cos(\omega_c-\omega_1)(t-t_1)$$
$$+ G\cos(\omega_c+\omega_2)(t-t_0) + G\cos(\omega_c-\omega_2)(t-t_0)$$

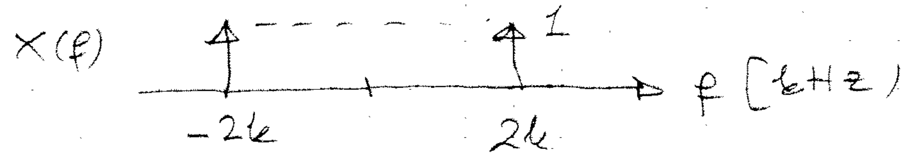
or

$$y(t) = GA_c\cos\omega_c(t-t_1) + G\cos\omega_1(t-t_1)\cos\omega_c(t-t_1)$$
$$+2G\cos\omega_2(t-t_0)\cos\omega_2(t-t_0)$$

$\Rightarrow \quad y(t) \neq \varphi_{AM}(t-t') \quad$ for some t'

$\Rightarrow$ phase distortion, $m(t)$ cannot be recovered from $y(t)$.

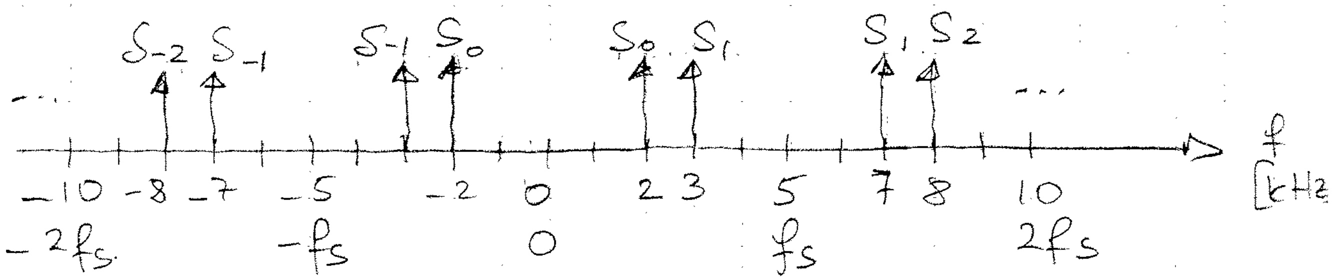
(2) Given that $x(t) = 2\cos\omega_0 t$ with $f_0 = 2$ kHz and



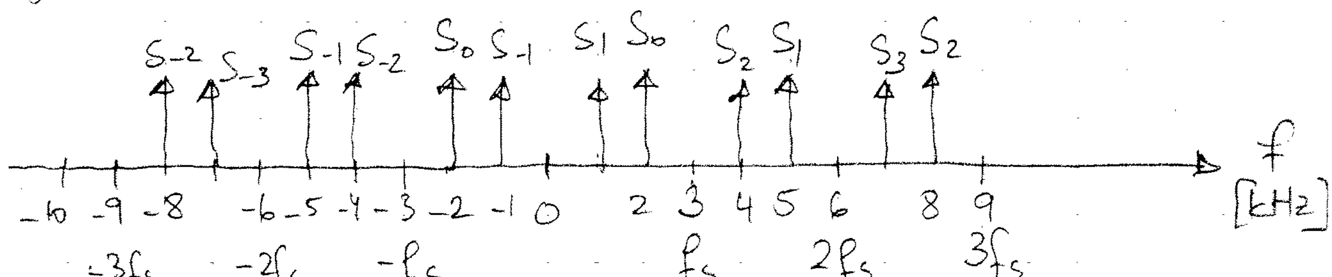
Since $x_s(t) = x(t)\,s(t)$

$$X_s(f) = X(f) * S(f)$$
$$= X(f) * \sum_n S_n \delta(f - nf_s)$$
$$= \sum_n S_n X(f - nf_s)$$

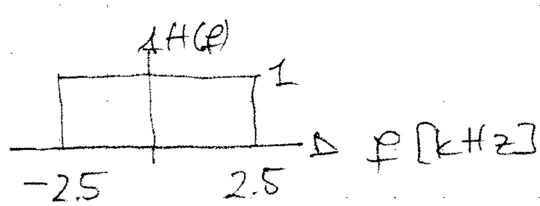
(a) $f_s = 5$ kHz



(b) $f_s = 3$ kHz



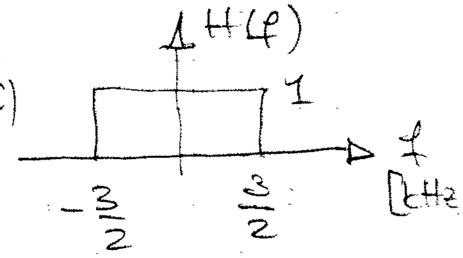
(c) If we use $H(f)$



$$\Rightarrow y(t) = S_0\, x(t) = S_0\, 2\cos\omega_0 t$$

$y(t)$ is a scaled version of $x(t)$, full recovery, since $f_s \geq 2f_0$

(d) Since $f_s = 3$ kHz, we now use $H(f)$



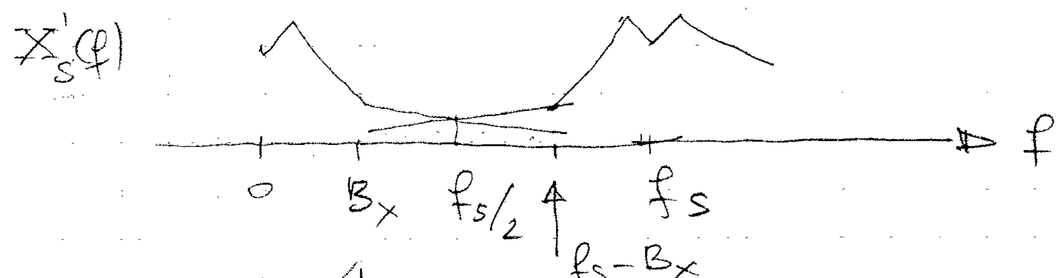
and the filter output $y(t)$ when $x_s(t)$ is the input becomes:

$y(t) = S_1 \, 2\cos(2\pi(1000)t)$ observe that $S_1 = S_{-1}$

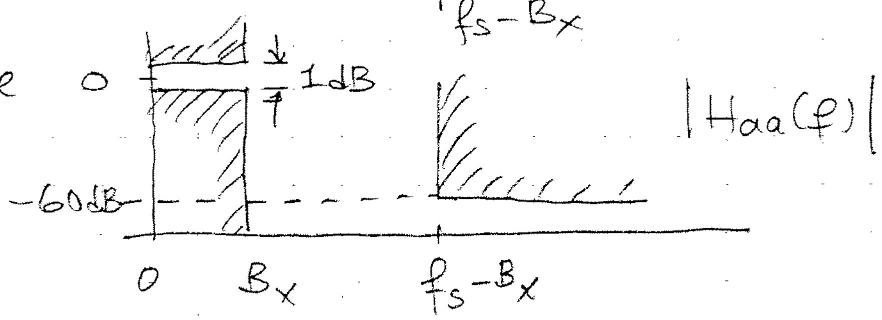
$\neq K x(t)$

Aliasing distortion, the input frequency component at f_0 is aliased to $f_s - f_0 = 3 - 2 = 1$ kHz.
In this case $f_s = 3$ kHz $< f_{Nyquist} = (2)(2)$ kHz.

(e) If we sample $x'(t)$ then the spectrum of $x'_s(t)$ will be



Hence



(3) Observe that we can write $\cos(\omega_0 t + \phi)$ as

$$\cos(\omega_0 t+\phi) = \frac{1}{2} e^{j\omega_0 t} \cdot e^{j\phi} + \frac{1}{2} e^{-j\omega_0 t} e^{-j\phi}$$

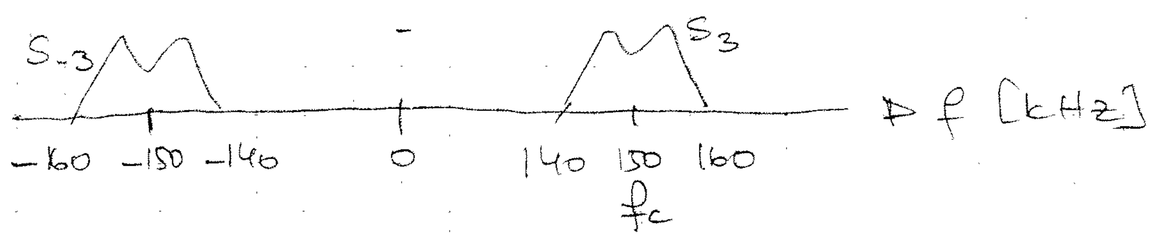
(a) Hence

$$\boxed{\mathcal{F}\left[m(t)\cos(\omega_0 t+\phi)\right] = \frac{1}{2}\left[M(f-f_0)e^{j\phi} + M(f+f_0)e^{-j\phi}\right]}$$

(b) We have $\mathcal{F}\left[m(t)\, s(t)\right] = \sum_n S_n M(f - nf_0)$,

with $f_0 = 50$ kHz and $B_m = 10$ kHz. Since we want to generate a DSB-SC signal with $f_c = 150$ kHz, we need to retain part of $m(t)s(t)$ that is centered at $\pm f_c$, which will correspond to the index $n = \pm 3$

The spectrum of the DSB-SC signal will be



i.e. $\Phi_{DSB\text{-}SC}(f) = S_3 M(f-3f_0) + S_{-3} M(f+3f_0)$

$$= K_3 M(f-3f_0)\, e^{j\phi_3} + K_3 M(f+3f_0)\, e^{-j\phi_3}$$

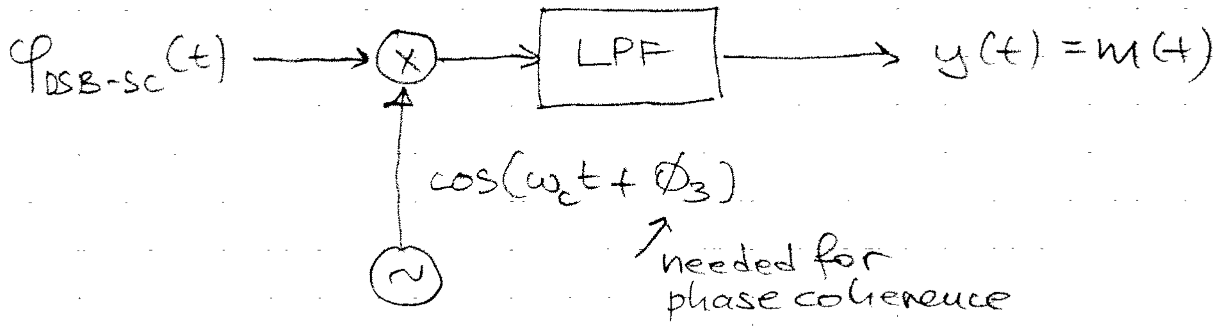
where we used the given information $S_n = S_{-n}^*$. Hence

$$S_3 = K_3 e^{j\phi_3} \quad \text{and} \quad S_{-3} = K_{-3} e^{+j\phi_{-3}} = K_3 e^{-j\phi_3}$$

Since, $3f_0 = f_c = 150$ kHz, we can write

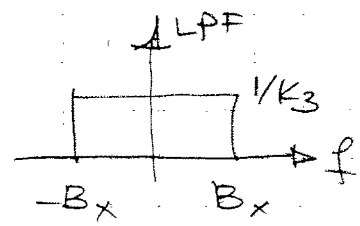
$$\boxed{\varphi_{DSB\text{-}SC}(t) = 2K_3\, m(t) \cos(2\pi f_c t + \phi_3)}$$

(c) Using the time-domain expression for $\varphi_{DSB-SC}(t)$ from in part (b), it is easy to design the coherent detector that we can use to recover $m(t)$ from $\varphi_{DSB-SC}(t)$:



with LPF: passband: $[0, B_x]$ Hz

passband gain: $1/K_3$.



Note: Passband gain equal $1/K_3$ as

$$y(t) = [2K_3 m(t) \cos(\omega_c t + \phi_3) \cos(\omega_c t + \phi_3)] * h_{LPF}$$
$$= [K_3 m(t) + K_3 \cos(2\omega_c t + 2\phi_3)] * h_{LPF}$$
$$= (K_3)(1/K_3)\, m(t).$$