Lecture Quiz 3, ENGR 233, Section Q, Solution

Problem. Use double integral to compute the area of the surface S that is the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

Formula:

$$A(S) = \iint_S dS = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA.$$

Solution. $f(x,y) = 1 - x^2 - y^2,\ f_x(x,y) = -2x,\ f_y(x,y) = -2y$

$$R = \{(x,y)\ :\ x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$$

Polar description of the region R: $0 \le \theta \le \pi/2,\ 0 \le r \le 1$. Using change to polar coordinates:

$$A(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{\pi/2} \int_0^1 r\sqrt{1 + 4r^2} dr d\theta$$

$$= \int_0^{\pi/2} 1 d\theta \int_0^1 r\sqrt{1 + 4r^2} dr\,.$$

$$v = 1 + 4r^2,\ r = 0 \to v = 1,\ r = 1 \to v = 5,\ dv = 8rdr,\ rdr = \frac{1}{8}dv.$$

$$A(S) = \left(\theta\Big|_0^{\pi/2}\right)\frac{1}{8}\int_1^5 \sqrt{v}dv = \frac{\pi}{16}\left(\frac{2}{3}v^{3/2}\Big|_1^5\right)$$

$$= \frac{\pi}{24}\left[5^{3/2} - 1\right] = \frac{\pi}{24}\left[5\sqrt{5} - 1\right].$$

Finally,

$$A(S) = \frac{\pi}{24}\left[5\sqrt{5} - 1\right].$$