APSC 172 **Assignment 4 Solutions** **Due March 16-17, 2015**

1. Using polar coordinates in the *x*-*y* plane, find the volume of the solid above the cone $z = \sqrt{3}r$ and below the hemisphere $z = \sqrt{4-r^2}$. As a check the answer is approximately 2.2448 but of course you have to calculate the exact answer.

Solution.
We need a base region in the *x*-*y* plane above which the solid lies. The cone intersects the hemisphere when $\sqrt{3}r = \sqrt{4-r^2}$ and this solves to give the circle $r = 1$. The solid is the region above $z_1 = \sqrt{3}r$ and below $z_2 = \sqrt{4-r^2}$ inside the circle $r \le 1$. Thus the volume is

$$V = \iint_R (z_2 - z_1)\,dA = \int_0^{2\pi}\int_0^1 \left(\sqrt{4-r^2} - \sqrt{3}r\right) r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 \left(r\sqrt{4-r^2} - \sqrt{3}r^2\right) dr\,d\theta$$

We calculate the two *r*-integrals separately. For the first we use the substitution $u = 4 - r^2$, $du = -2r\,dr$

$$\int_0^1 r\sqrt{4-r^2}\,dr = \int_4^3 -\sqrt{u}\,\frac{du}{2} = -\frac{1}{3}u^{3/2}\Big]_4^3 = -\frac{1}{3}\left(3^{3/2} - 4^{3/2}\right) = \frac{1}{3}\left(8 - 3\sqrt{3}\right) == \frac{8}{3} - \sqrt{3}\ .$$

The second is

$$\sqrt{3}\int_0^1 r^2\,dr = \sqrt{3}\frac{r^3}{3}\Big]_0^1 = \frac{\sqrt{3}}{3}.$$

Subtract the second from the first:

$$\int_0^1 \left(r\sqrt{4-r^2} - \sqrt{3}r^2\right) dr = \frac{8}{3} - \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{8 - 4\sqrt{3}}{3}\ .$$

Then:

$$\int_0^{2\pi}\int_0^1 \left(r\sqrt{4-r^2} - \sqrt{3}r^2\right) dr\,d\theta = \frac{8 - 4\sqrt{3}}{3}(2\pi) = \frac{16 - 8\sqrt{3}}{3}\pi \approx 2.2448.$$

2. A circular pool with radius 20 has water depth 2 at the south end and depth 7 at the north end and in between increases linearly in the N-S direction. In the E-W direction, the depth is always constant. Find the volume of water in the pool in two different ways.

(a) by calculating the integral of the depth. Use polar coordinates in the *x*-*y* plane.

(b) by using symmetry to find the average depth of the pool. Take care with your argument.

Solution:

(a) If the pool is centred at the origin with N-S along the *y*-axis, the depth *z* is linear in *y* with slope 5/40 = 1/8. Thus:

$$z = 2 + (y+20)/8 = \frac{y+36}{8}.$$

We want to integrate this on the disc centred at the origin with radius 20:

$$V = \iint_R \frac{y+36}{8}\, dA = \int_0^{2\pi}\int_0^{20} \frac{r\sin\theta + 36}{8}\, r\,dr\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\int_0^{20} (r^2 \sin\theta + 36r)\,dr\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{r^3}{3}\sin\theta + 18r^2\right]_{r=0}^{20}$$

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{8000}{3}\sin\theta + 7200\right]d\theta$$

$$= \left[-\frac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi}$$

$$= 0 + 1800\pi = 1800\pi.$$

(b) The depth of the pool is independent of *x* and is 9/2 along the *x*-axis. It is also antisymmetric with respect to the *x*-axis, that is the depth at (x, y) and the depth at $(x, -y)$ average to give 9/2. Thus the average depth of the entire pool is 9/2. Now volume is area×avge depth:

$$V = 20^2\pi \times 9/2 = 1800\pi.$$

3. Find the mass and centre of mass of the solid S below the paraboloid $z = 4 - (x^2 + y^2)$ and above the plane $z = 0$ if the density is ρ at any point (x, y, z) is the distance of the point from the z-axis. Use cylindrical coordinates (polar in the x-y plane). As a check the mass is approximately 26.81 but of course you have to calculate the exact mass.

Solution

The base of the solid is formed by the intersection of the paraboloid with the plane $z = 0$ and that's the circle $x^2 + y^2 = 4$ of radius 2. The density is the distance from the z-axis and that's r.

The mass is $M = \iiint_S \rho(x, y, z)dV = \iiint_S rdV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} rdzrdrd\theta$

$$= \int_0^{2\pi}\int_0^2 rz\Big|_{z=0}^{4-r^2} rdrd\theta = \int_0^{2\pi}\int_0^2 r(4-r^2)rdrd\theta = \int_0^{2\pi}\int_0^2 \left(4r^2 - r^4\right)drd\theta$$

$$= \int_0^{2\pi}\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 d\theta = \int_0^{2\pi}\left[\frac{32}{3} - \frac{32}{5}\right]d\theta = \int_0^{2\pi}\frac{64}{15}d\theta = \frac{128\pi}{15}.$$

Now for the centre of mass. By symmetry it will be on the z-axis. Let it be at $\bar{z}$. Then:

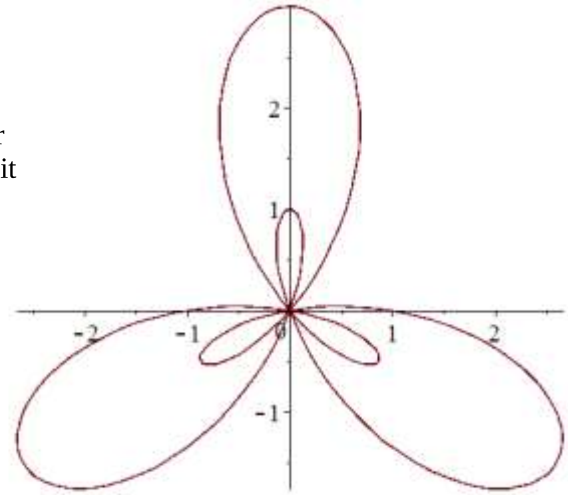
$$M\bar{z} = \iiint_S z\rho(x, y, z)dV = \iiint_S zrdV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} rzdzrdrd\theta$$

$$= \int_0^{2\pi}\int_0^2 r\frac{z^2}{2}\Big|_{z=0}^{4-r^2} rdrd\theta = \int_0^{2\pi}\int_0^2 r\frac{(4-r^2)^2}{2}rdrd\theta = \int_0^{2\pi}\int_0^2 \frac{16r^2 - 8r^4 + r^6}{2}drd\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{16r^3}{3} - \frac{8r^5}{5} + \frac{r^7}{7}\right]_0^2 d\theta = \int_0^{2\pi}\left[\frac{64}{3} - \frac{128}{5} + \frac{64}{7}\right]d\theta = \int_0^{2\pi}\frac{512}{105}d\theta = \frac{1024\pi}{105}.$$

Thus $\bar{z} = \frac{1}{M}\frac{1024\pi}{105} = \frac{15}{128\pi}\frac{1024\pi}{105} = \frac{8}{7}$.

4. At the right is the graph of the double 3-leafed rose
$r = 1 - 2\sin(3\theta)$.
Right away you might wonder how there can be two r values for a given θ, but when I asked Maple to plot this curve that's what it gave me. For example, for $\theta = \pi/2$, the diagram tells me that there are two r values, 1 and 3.



(a) What's going on here? How there can be two r values?
(b) Calculate the areas of the big leaf and the small leaf.
As a check the answers are 2.9604 and 0.1812 (so that the area *between* the leaves will be the difference 2.7792). These are approximate but of course you have to calculate the exact answers.

[Tech help: $\int \sin^2(k\theta)d\theta = \frac{1}{2}\theta - \frac{1}{4k}\sin(2k\theta) + C$.]

Solution.
(a) For $\theta = \pi/2$, the formula gives me $r = 1 - 2\sin(3\pi/2) = 1+2 = 3$ and that's the intersection at (0, 3). The intersection at (0, 1) comes from putting $\theta = 3\pi/2$, the formula gives me

$r = 1 - 2\sin(9\pi/2) = 1 - 2\sin(\pi/2) = 1-2 = -1$

and the convention here is that when the formula produces a negative r, the point is projected an equal distance through the origin. Thus the point (0, –1) for $\theta = 3\pi/2$ is projected through to the other side of the origin giving us the point at (0, 1).

(b) The discussion of (a) gives us the θ limits to use for the areas of the two leaves, the big and the small, that lie along the y-axis. We need to keep in mind that the big leaf actually does lie along the positive y-axis whereas the small leaf really lies along the negative y-axis but the r-convention has projected it through the origin.

Now what are the limits of integration going to be for these two areas? Well they are the angles that are tangent to the leaf at the origin and these will be the angles at which r takes the value 0. To find these set:

$1 - 2\sin(3\theta) = 0$

$\sin(3\theta) = ½$

$3\theta = 30, 150, 390, 510, 750, 870, \ldots$

$\theta = 10, 50, 130, 170, 250, 290, \ldots$

Here, I have used degree measure as that makes it easier for me to relate to the diagram and I have also used large enough values for 3θ so I can get the limits for the small vertical leaf which lies either side of 270°. Thus the limits of the large vertical leaf are 50° and 130° (so that it is centred at 90° and is 80° wide) and the limits of the small vertical leaf are 250° and 290° (so that it is centred at 270° and is 40° wide—that's before it is projected through the origin).

In radian measure the θ-angles are

$\theta = \pi/18, 5\pi/18, 13\pi/18, 17\pi/18, 25\pi/18, 29\pi/18, \ldots$

Thus the limits of the large vertical leaf are $5\pi/18$ and $13\pi/18$ and the limits of the small vertical leaf are $25\pi/18$ and $29\pi/18$.

The calculations for the large leaf are:

$$A = \iint_R 1 dA = \int_{5\pi/18}^{13\pi/18} \int_0^{1-2\sin 3\theta} r dr d\theta = 2\int_{5\pi/18}^{\pi/2} \int_0^{1-2\sin 3\theta} r dr d\theta \quad \text{(by symmetry).}$$

$$= 2\int_{5\pi/18}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{1-2\sin 3\theta} d\theta = 2\int_{5\pi/18}^{\pi/2} \frac{(1-2\sin 3\theta)^2}{2} d\theta = \int_{5\pi/18}^{\pi/2} \left(1 - 4\sin 3\theta + 4\sin^2 3\theta\right) d\theta$$

$$= \theta + \frac{4}{3}\cos 3\theta + 4\left(\frac{1}{2}\theta - \frac{1}{12}\sin(6\theta)\right)\Bigg|_{\theta=5\pi/18}^{\pi/2} = 3\theta + \frac{4}{3}\cos 3\theta - \frac{1}{3}\sin(6\theta)\Bigg|_{\theta=5\pi/18}^{\pi/2}$$

$$= 3\frac{\pi}{2} + \frac{4}{3}\cos(3\pi/2) - \frac{1}{3}\sin(3\pi) - \left(\frac{5\pi}{6} + \frac{4}{3}\cos(5\pi/6) - \frac{1}{3}\sin(5\pi/3)\right)$$

$$= 3\frac{\pi}{2} + \frac{4}{3}(0) - \frac{1}{3}(0) - \left(\frac{5\pi}{6} + \frac{4}{3}\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{3}\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{2\pi}{3} + \frac{\sqrt{3}}{2} \approx 2.9604$$

The calculations for the small leaf are:

$$A = \iint_R 1 dA = \int_{25\pi/18}^{29\pi/18} \int_0^{1-2\sin 3\theta} r dr d\theta = 2\int_{25\pi/18}^{3\pi/2} \int_0^{1-2\sin 3\theta} r dr d\theta \quad \text{(by symmetry).}$$

$$= 2\int_{25\pi/18}^{3\pi/2} \left[\frac{r^2}{2}\right]_0^{1-2\sin 3\theta} d\theta = 2\int_{25\pi/18}^{3\pi/2} \frac{(1-2\sin 3\theta)^2}{2} d\theta = \int_{25\pi/18}^{3\pi/2} \left(1 - 4\sin 3\theta + 4\sin^2 3\theta\right) d\theta$$

$$= \theta + \frac{4}{3}\cos 3\theta + 4\left(\frac{1}{2}\theta - \frac{1}{12}\sin(6\theta)\right)\Bigg|_{\theta=25\pi/18}^{3\pi/2} = 3\theta + \frac{4}{3}\cos 3\theta - \frac{1}{3}\sin(6\theta)\Bigg|_{\theta=25\pi/18}^{3\pi/2}$$

$$= 3\frac{3\pi}{2} + \frac{4}{3}\cos(9\pi/2) - \frac{1}{3}\sin(9\pi) - \left(\frac{25\pi}{6} + \frac{4}{3}\cos(25\pi/6) - \frac{1}{3}\sin(25\pi/3)\right)$$

$$\frac{9\pi}{2} + \frac{4}{3}(0) - \frac{1}{3}(0) - \left(\frac{25\pi}{6} + \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} \approx 0.1812$$