MATH 105 202 Midterm 1 Solutions

1. (a) Let

$$f(x, y) = \sqrt{x^2 + 2y^2}$$

Use $f_x(1, 2)$ and $f_y(1, 2)$ to estimate $f(1.3, 1.7)$.

Solution: Calculate the first-order partial derivatives of f,

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + 2y^2}} \qquad f_y(x, y) = \frac{2y}{\sqrt{x^2 + 2y^2}}$$

Since $f_x(x, y)$ and $f_y(x, y)$ are continuous at the point $(1, 2)$ (in fact, they are continuous everywhere except $(0, 0)$), it follows from the Condition of Differentiability that f is differentiable at $(1, 2)$ (similarly, for all points except $(0, 0)$). Recall that f being differentiable at $(1, 2)$ means that as (x, y) varies from $(1, 2)$ to $(1 + \Delta x, 2 + \Delta y)$, the change in the value of f can be well-approximated by $f_x(1, 2)\Delta(x) + f_y(1, 2)\Delta(y)$. Thus, to estimate $f(1.3, 1.7)$, we first calculate the following:

$$\Delta x = 1.3 - 1 = 0.3 \qquad \Delta y = 1.7 - 2 = -0.3$$
$$f_x(1, 2) = \frac{1}{\sqrt{1^2 + 2(2^2)}} = \frac{1}{3} \qquad f_y(1, 2) = \frac{4}{\sqrt{1^2 + 2(2^2)}} = \frac{4}{3}$$
$$f(1, 2) = \sqrt{1^2 + 2(2^2)} = 3$$

Then, it follows from f being differentiable at $(1, 2)$ that $f(1.3, 1.7)$ can be estimated by:

$$\begin{aligned} f(1.3, 1.7) &\approx f(1, 2) + f_x(1, 2)\Delta x + f_y(1, 2)\Delta y \\ &\approx 3 + \frac{1}{3}(0.3) + \frac{4}{3}(-0.3) \approx 2.7 \end{aligned}$$

(b) Let $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle -b, a\rangle$, where a, b are arbitrary real numbers, at least one of which is non-zero. Are the two vectors $\mathbf{v}$ and $\mathbf{w}$ always perpendicular, irrespective of the choices of a and b?

Solution: Recall that given two vectors in two-dimensional space $\mathbf{u} = \langle u_1, u_2\rangle$, and $\mathbf{v} = \langle v_1, v_2\rangle$, then $\mathbf{u} \perp \mathbf{v}$ if $u_1v_1 + u_2v_2 = 0$. Thus, for $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle -b, a\rangle$, we have that $a(-b) + ba = 0$ for any choices of a and b. So, the two vectors $\mathbf{v}$ and $\mathbf{w}$ are always perpendicular, irrespective of the choices of a and b.

(c) Find a unit vector parallel to $\langle 3, \sqrt{8}, -\sqrt{8}\rangle$.

Solution: Recall that given a non-zero vector $\mathbf{v}$, then $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector parallel to $\mathbf{v}$. So, for $\mathbf{v} = \langle 3, \sqrt{8}, -\sqrt{8}\rangle$, we get:

$$|\mathbf{v}| = \sqrt{3^2 + (\sqrt{8})^2 + (-\sqrt{8})^2} = \sqrt{25} = 5$$

So, a unit vector that is parallel to $\langle 3, \sqrt{8}, -\sqrt{8}\rangle$ is $\left\langle \dfrac{3}{5}, \dfrac{\sqrt{8}}{5}, \dfrac{-\sqrt{8}}{5} \right\rangle$.

(d) Compute the left Riemann sum with three equal subintervals for $f(x) = \dfrac{15}{x}$ in the interval $[1, 7]$.

Solution: For $n = 3$, $f(x) = \dfrac{15}{x}$ and $a = 1, b = 7$, we have that $\Delta x = \dfrac{b-a}{n} = \dfrac{7-1}{3} = 2$. The points $\overline{x_k}$ for the left Riemann sum are given by: $\overline{x_k} = a + (k-1)\Delta x = 1 + (k-1)2 = 2k - 1$. Thus, the left Riemann sum of f using 3 subintervals is:

$$\sum_{k=1}^{3} f(\overline{x_k})\Delta x = \sum_{k=1}^{3} \frac{30}{2k-1} = 30 + 10 + 6 = 46.$$

(e) Is there a function $f(x, y)$ such that $\nabla f(x, y) = \langle 2xy, x^2 + xy\rangle$? If not, explain why no such function exists; otherwise find $f(x, y)$. State clearly any result that you use.

Solution: Suppose that $f(x, y)$ is a function such that $\nabla f(x, y) = \langle 2xy, x^2 + xy\rangle$. Then,

$$f_{xy} = 2x \neq 2x + y = f_{yx}.$$

Note that f_{xy} and f_{yx} are both continuous on $\mathbb{R}^2$, being polynomials. So, $f_{yx} \neq f_{xy}$ contradicts Clairaut's Theorem which states that if f_{yx} and f_{xy} are continuous, then $f_{xy} = f_{yx}$. Therefore, there does not exist a function $f(x, y)$ such that $\nabla f(x, y) = \langle 2xy, x^2 + xy\rangle$.

(f) Let R be the semicircular region $\{x^2 + y^2 \leq 4, x \leq 0\}$. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 + 2x$$

on the *boundary of the region R*.

Solution: The boundary of the region R consists of two pieces: the semicircular arc which can be parametrized by $x = 2\cos\theta$ and $y = 2\sin\theta$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, and the vertical segment $x = 0$ for $-2 \le y \le 2$. We will find the potential candidates where the maximum and minimum can occur on each piece:

- *On the semicircular arc*: We have that $f(x, y) = g(\theta) = (2\cos\theta)^2 + (2\sin\theta)^2 + 2(2\cos\theta) = 4 + 4\cos\theta$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$. Then, $g'(\theta) = -4\sin\theta = 0$ if and only if $\theta = \pi$. So, there are 3 points where extrema can occur: $(-2, 0)$ (critical point), $(0, -2)$ and $(0, 2)$ (end points).
- *On the vertical segment*: We have that $f(0, y) = h(y) = y^2$ for $-2 \le y \le 2$. So, $h'(y) = 2y = 0$ when $y = 0$. So, there are 3 points where extrema can occur: $(0, 0)$ (critical point), $(0, -2)$ and $(0, 2)$ (end points).

Evaluate f at those points, we get:

$$f(-2, 0) = 0, \quad f(0, -2) = 4, \quad f(0, 2) = 4, \quad f(0, 0) = 0$$

Thus, on the boundary of R, f attains the absolute maximum value 4 at the points $(0, -2)$ and $(0, 2)$ and the absolute minimum value 0 at the points $(-2, 0)$ and $(0, 0)$.

2. Find *all* critical points of the function:

$$f(x, y) = \frac{x^3}{3} - \frac{y^3}{3} + 3xy$$

Classify each point as a local minimum, local maximum, or saddle point.

Solution: Observe that f is defined at every point in $\mathbb{R}^2$. Compute the first-order partial derivatives of f:

$$f_x(x, y) = x^2 + 3y \qquad f_y(x, y) = -y^2 + 3x$$

Since both f_x and f_y are defined at every point in $\mathbb{R}^2$, the only critical points of f are those at which $f_x = f_y = 0$. Solving $f_x = 0$ for y, we get $y = \frac{-x^2}{3}$. Substitute that into $f_y = 0$, we get:

$$\begin{aligned} 0 &= -y^2 + 3x = -\left(\frac{-x^2}{3}\right)^2 + 3x = \frac{-x^4 + 27x}{9} \\ &= x(27 - x^3) \\ &\Rightarrow x = 0, x = 3 \end{aligned}$$

Using $y = \dfrac{-x^2}{3}$, we get that when $x = 0$, then $y = 0$, and when $x = 3$ then $y = -3$. So, there are two critical points $(0, 0)$ and $(3, -3)$. Compute the second-order partial derivatives and the discriminants,

$$f_{xx} = 2x, \quad f_{yy} = -2y, \quad f_{xy} = 3, \quad D(x, y) = -4xy - 9$$

Using the Second Derivative Test to classify the points, we get:

- At the point $(0, 0)$, $D(0, 0) = -9 < 0$, so $(0, 0)$ is a saddle point.
- At the point $(3, -3)$, $D(3, -3) = 27 > 0$ and $f_{xx}(3, -3) = 6 > 0$, so $(3, -3)$ is a local minimum.

3. A cookie company produces sugar-free cookies, where each cookie contains s milligrams of sucralose and t milligrams of acesulfame. Market studies show that consumer satisfaction is best if the sweetness level of the cookie is maintained according to the equation:

$$\sqrt{s} + \sqrt{t} = 5$$

If the price of sucralose is \$20 per gram and the price of acesulfame is only \$5 per gram, use Lagrange multipliers to find how the company should choose s and t so as to minimize costs. (Remember, that was 1000 milligrams in one gram).

Clearly state the objective function and the constraint. There is no need to justify that the solution you obtained is the absolute max or min. **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.**

Solution: Since acesulfame costs \$5 per gram, that means it costs \$0.02 per milligram. Similarly, we get \$0.005 per milligram. So, the cost function which is the objective function is: $f(s, t) = 0.02s + 0.005t$ and the constraint is $g(s, t) = \sqrt{s} + \sqrt{t} - 5 = 0$. Using Lagrange multiplier, we need to solve the following system of equations:

$$\nabla f(s,t) = \lambda \nabla g(s,t)$$
$$g(s,t) = 0$$

More explicitly, we need to solve:

$$0.02 = \frac{\lambda}{2\sqrt{s}}$$
$$0.005 = \frac{\lambda}{2\sqrt{t}}$$
$$\sqrt{s} + \sqrt{t} - 5 = 0$$

From the first and second equations, we get $\sqrt{s} = 25\lambda$ and $\sqrt{t} = 100\lambda$. Substitute into the third equation, we get:

$$25\lambda + 100\lambda = 5$$
$$\lambda = \frac{1}{25}$$

Thus, $\sqrt{s} = 25\left(\frac{1}{25}\right) = 1$ and $s = 1$. Similarly, $\sqrt{t} = 100\left(\frac{1}{25}\right) = 4$, so $t = 16$. Therefore, if the company chooses to make cookies such that each contains 1 milligram of sucralose and 16 milligrams of acesulfame, then that would minimize the cost.

4. Consider the surface S given by:

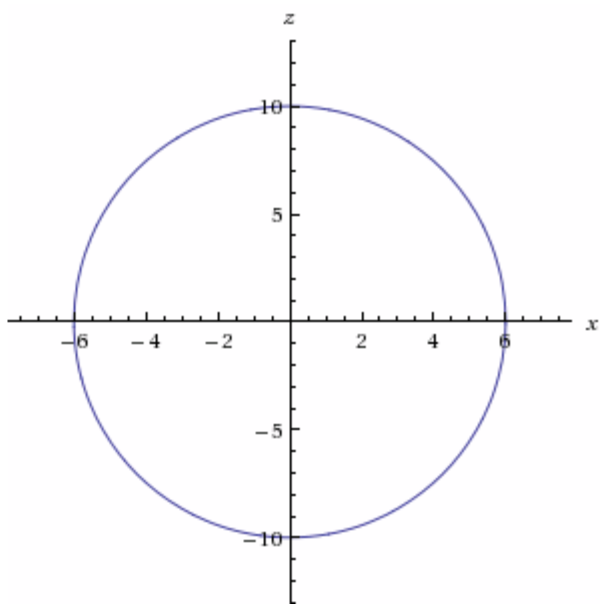
$$y - \frac{x^2}{9} = \frac{z^2}{25}$$

(a) Sketch the traces of S in the $y = 4$ and $z = 0$ planes.

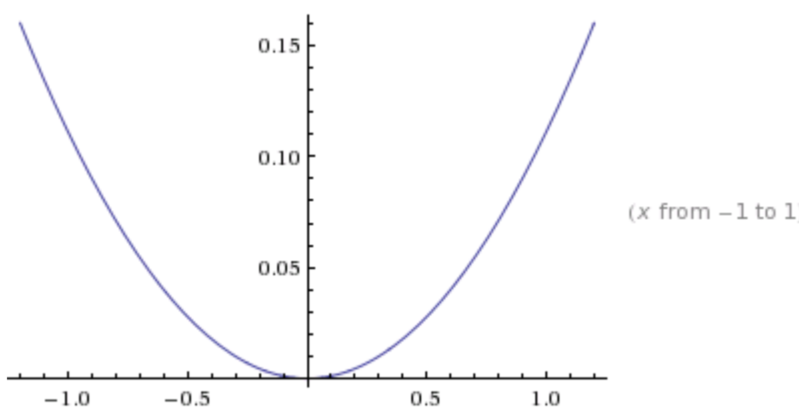
Solution:

- For $y = 4$: The equation for the trace is $4 - \frac{x^2}{9} = \frac{z^2}{25}$, which is equivalent to $\frac{x^2}{36} + \frac{z^2}{100} = 1$. Thus, in the xz-plane, the graph of the trace is an ellipse centered at $(0,0)$ with x-intercepts $(6,0)$ and $(-6,0)$, and z-intercepts

$(0, 10)$ and $(0, -10)$.



- For $z = 0$: The equation for the trace is $y - \frac{x^2}{9} = 0$, which is equivalent to $y = \frac{x^2}{9}$. Thus, in the xy-plane, the graph of the trace is a parabola with vertex at $(0, 0)$ being compressed vertically.



(b) Based on the traces you sketched above, which of the following renderings represents the graph of the surface?

Solution: The answer is C. To exclude other possibilities, we first look at the trace in the $z = 0$ plane, which should be a parabola as calculated above.

Notice that the graphs in A and B have only one point $(0, 0, 0)$ as its trace in the $z = 0$ plane, and the graphs in E and F have no points as its trace in the $z = 0$ plane. So, we can exclude those, and consider only the remaining two possibilities C and D. We now look at the trace in the $y = 4$ plane, which should be a parabola symmetric with respect to the y-axis as calculated above. This yields that C is the unique graph with such property, as the trace in $y = 4$ plane of the graph in D is a parabola symmetric with respect to x-axis (not y-axis).

(c) (Extra credit) Transform the limit of the following Riemann sum to a definite integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n} \cos\left(1 + \frac{4k}{n}\right)$$

Solution: Suppose we want to identify the sum as the right Riemann sum with left endpoint $a = 0$. Since $\Delta x = \frac{2}{n} = \frac{b-0}{n}$, we get that $b = 2$. Then, since we use the right Riemann sum, the point $\overline{x_k}$ is given by: $\overline{x_k} = a + k\Delta x = \frac{2k}{n}$. Thus, it follows from

$$f(\overline{x_k}) = \cos\left(1 + \frac{4k}{n}\right) = \cos(1 + 2\overline{x_k})$$

that $f(x) = \cos(1 + 2x)$. So, the integral that the above Riemann sum represents is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n} \cos\left(1 + \frac{4k}{n}\right) = \int_0^2 \cos(1 + 2x)\, dx$$

Remark: The answer in this question is not unique, as we may represent the same integral in many different ways. If we actually evaluate the integral, then we should get the same value regardless of how we represent it.