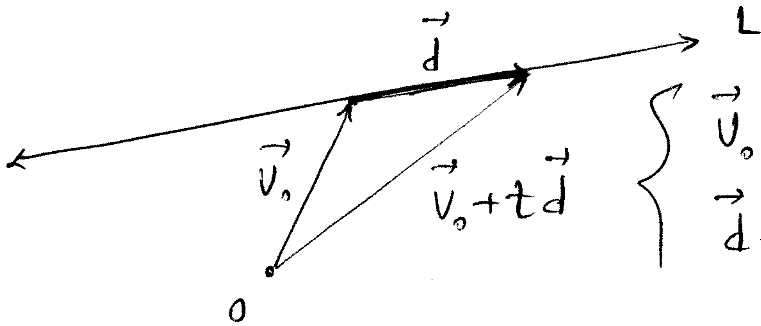
CHAPTER 3 : "Lines & planes" (Review)

3.1 Describing Lines:

vector parametric construction of line "L":

$$L = \{ \vec{V}_0 + t\vec{d} \mid t \in \mathbb{R} \},$$



$$\begin{cases} \vec{V}_0 : \text{position vector} \\ \vec{d} \neq \vec{0} : \text{direction vector} \end{cases}$$

or, any point $\vec{V}$ on the line "L" can be written as $\vec{V} = \vec{V}_0 + t\vec{d}$.

example : $y = 3x + 2$ in $\mathbb{R}^2$. parametric equation : put $x = t$ and $y = 3t + 2$, or in the vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 3t+2 \end{pmatrix} = \begin{pmatrix} 0 + 1t \\ 2 + 3t \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow L = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

Note : • parametric equations of lines are not unique!

• The form $L = \{ \vec{V}_0 + t\vec{d} \mid t \in \mathbb{R} \}$ for a line in $\mathbb{R}^n$ is called the "vector form" or "parametric form".

• we can write parametric forms in terms of coordinates: if $\vec{V}_0 = (a, b, c)$ and $\vec{d} = (d_1, d_2, d_3)$, then $\vec{V} = \vec{V}_0 + td$ is $\vec{V} = (x_1, x_2, x_3)$, where

$$\begin{cases} x_1 = a + td_1 \\ x_2 = b + td_2 \\ x_3 = c + td_3 \end{cases}, \quad t \in \mathbb{R}$$

Note: The reason for non-uniqueness of parametric forms of a Line "L" is that any point on "L" will work as a position vector, and any non-zero scalar multiple of a direction vector is a vector in the same direction.

Example:

$$L = \{(4,3,-1) + t(1,1,2) \mid t \in \mathbb{R}\}, \ \vec{U}_0 = (4,3,-1), \ \vec{d} = (1,1,2)$$

$$L' = \{(5,4,1) + t(-2,-2,-4) \mid t \in \mathbb{R}\}, \ \vec{U}_0 = (5,4,1), \ \vec{d} = (-2,-2,-4)$$

L and L' are the same line.

Example: consider the following lines:

$$L = \{(0,1,7) + t(1,-1,1) \mid t \in \mathbb{R}\}$$

$$M = \{(12,0,0) + t(-3,3,-3) \mid t \in \mathbb{R}\}$$

$$N = \{(0,6,0) + t(2,0,-2) \mid t \in \mathbb{R}\}$$

Are "L" & "M" parallel? Orthogonal? Do they intersect? Same questions for other lines.

Solution:
- $(-3,3,-3) = -3.(1,-1,1)$ Thus L & M are parallel, but not orthogonal and they don't intersect, as $(12,0,0)$ is not in L.
- L and N are orthogonal, because $(+1,-1,1).(2,0,-2) = 0$
- M and N are orthogonal, because $(-3,3,-3).(2,0,-2) = 0$
- M and N intersect in the point $(6,6,-6)$. (Check!)

3.3 Planes in $\mathbb{R}^3$:

There exist three ways to represent them:

(1) "parametric form":

parameters ($t, s \in \mathbb{R}$)

$$(x, y, z) = (a_0, b_0, c_0) + t(a_1, b_1, c_1) + s(a_2, b_2, c_2)$$

position vector direction vectors

(2) "Cartesian form": a plane in $\mathbb{R}^3$ is described as the set of all points (x, y, z), such that:

$$ax + by + cz = d, \quad (a, b, c, d \in \mathbb{R})$$

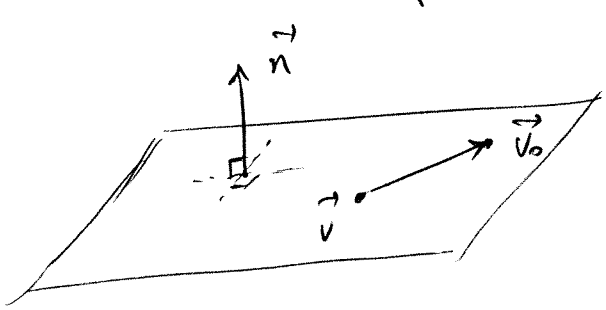
where $\vec{n} = (a, b, c)$ is a "normal vector" to the plane.

(3) "Normal form": a plane "W" in $\mathbb{R}^3$ through $\vec{v}_0$ with normal vector $\vec{n}$ is:

$$W = \{ \vec{v} \in \mathbb{R}^3 \mid (\vec{v} - \vec{v}_0) \cdot \vec{n} = 0 \},$$

where $\vec{v} = (x, y, z)$ represents points on the plane W and $\vec{v}_0$ is the position vector.

<u>Note</u>: the normal vector is a non-zero vector and is orthogonal to all vectors in the plane.



example: $\vec{v}_0 = (1,0,3)$, $\vec{n} = (-1,1,2)$, then the plane W through $\vec{v}_0$ with normal vector $\vec{n}$ is:

$$W = \{ \vec{v} \in \mathbb{R}^3 \mid (\vec{v} - \vec{v}_0) \cdot \vec{n} = 0 \}$$
$$= \{ (x,y,z) \in \mathbb{R}^3 \mid ((x,y,z) - (1,0,3)) \cdot (-1,1,2) = 0 \}$$
$$= \{ (x,y,z) \in \mathbb{R}^3 \mid -(x-1) + (y-0) + 2(z-3) = 0 \}$$
$$= \{ (x,y,z) \in \mathbb{R}^3 \mid -x + y + 2z = 5 \}$$

example: Find the intersection of the plane $3x - z = 10$, and the line $(1,9,1) + t(1,6,2)$:

solution: plug in $x = 1+t$, $y = 9+6t$, $z = 1+2t$ into the equation of the plane:

$$3(1+t) - (1+2t) = 10 \Leftrightarrow 3 + 3t - 1 + 2t = 10 \Leftrightarrow$$
$$2 + t = 10 \Leftrightarrow t = 8$$

Then replace $t = 8$ into the equation of the line, namely

$$(x,y,z) = (1,9,1) + 8(1,6,2) = (9,57,17) \checkmark$$

example: Find the intersection of two planes $x+y+z=3$, $x-y-z=2$.

solution: subtract the second equation from the first;

$$2y + 2z = 1 \Leftrightarrow y = \frac{1}{2} - z$$

plug in $y = \frac{1}{2} - z$ into the equation of the first plane:

$$x + (\tfrac{1}{2} - z) + z = 3 \Leftrightarrow x = 5/2$$

Then, reparametrize $x = \frac{5}{2}$, $y = \frac{1}{2} - t$, $z = t$:

$$L = \{ (\tfrac{5}{2}, \tfrac{1}{2}, 0) + t(0,-1,1) \mid t \in \mathbb{R} \} \checkmark$$

3.4 Geometry of Planes in $\mathbb{R}^n$:

n	Equation in $\mathbb{R}^n$	Resulting Geometric Object
1	$ax = b$	Point
2	$ax + by = c$	Line
3	$ax + by + cz = d$	Plane
4	$ax_1 + bx_2 + cx_3 + dx_4 = e$	?

3.5 Cross Products in $\mathbb{R}^3$:

Notation: given $a, b, c, d \in \mathbb{R}$, let $\begin{vmatrix} a & b \\ c & d \end{vmatrix} :=_{def} ad - bc$,

(This is a determinant; more on this later.)

Given $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$, then

$$\vec{u} \times \vec{v} := \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$$

Example: $\vec{u} = (3, 1, 4)$, $\vec{v} = (-2, 1, -1)$, then

$$\vec{u} \times \vec{v} = \left(\begin{vmatrix} 1 & 4 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix}, \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} \right) = (-5, -5, 5)$$

- Another definition based on unit vectors $\hat{i}, \hat{j}, \hat{k}$:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \hat{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \hat{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \hat{k}.$$

- properties of the cross product:

(1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

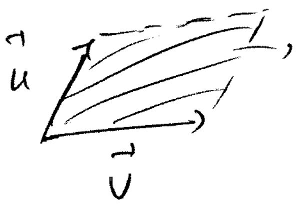
(2) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$

(3) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$

$\Big\}$: $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ & $\vec{v}$.

(4) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

(5) $\| \vec{u} \times \vec{v} \| = \| \vec{u} \| \cdot \| \vec{v} \| \cdot \sin(\theta)$, where $0 \leq \theta \leq \pi$ is the angle between $\vec{u}$ and $\vec{v}$. This is in fact the area of the parallelogram with sides $\vec{u}$ & $\vec{v}$.



, area of parallelogram $= \| \vec{u} \times \vec{v} \|$

(6) $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$ in general

Application (I): Normal vectors:

The cross product gives a normal vector to the plane parallel to direction vectors $\vec{u}$ & $\vec{v}$.

example: Find an equation of the plane containing the three points $A=(1,2,3)$, $B=(1,0,0)$, $C=(0,1,1)$:

$\vec{BA} = (0,2,3)$, $\vec{BC} = (-1,1,1)$ are parallel to the plane, so

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 3 \\ -1 & 1 & 1 \end{vmatrix} = (-1,-3,2).$$

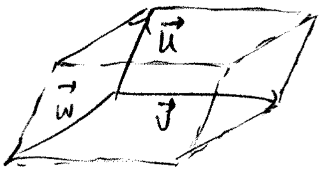
So the equation for the plane has the form $-x-3y+2z=d$, where "d" can be found by plugging in one of the points; then $d=-1$.

- Application (II): Volume of a parallelepiped.

Given $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, the volume of a parallelepiped with sides $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, is given by

$$|(\vec{u} \times \vec{v}) \cdot \vec{w}|.$$

(The order of vectors is not important)



<u>example</u>: $\vec{u} = (1,1,2)$, $\vec{v} = (3,0,1)$, $\vec{w} = (4,1,1)$

$$\Rightarrow \text{Volume} = \left| (1,1,2) \cdot \left(\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} \right) \right|$$

$$= |(1,1,2) \cdot (-1,1,3)| = |-1+1+6| = 6.$$

<u>Further remarks</u>:

(1) $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times \vec{v}) = (\vec{w} \times \vec{u}) \cdot \vec{v}$ (etc)

(with the same <u>cyclic order</u>)

(2) Three vectors $\vec{u}, \vec{v}, \vec{w}$ are coplanar $\Leftrightarrow$ $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

(3) $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$, $\vec{w} = (x_3, y_3, z_3)$, then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

: 3x3 determinant (Later).