

MAT2384-Ordinary differential Equations and Numerical Methods-Fall 2010

MIDTERM EXAM (October 22, 2010)

Professor: Joseph Houry

Duration: 80 minutes

Last Name: Solution

First Name: _____

Student Number: _____

- (1) This is a closed book exam.
- (2) Only basic scientific calculators are allowed. Graphing or programmable calculators are not permitted.
- (3) The exam has 6 questions worth a total of 30 points.
- (4) The exam has 9 pages.
- (5) Please write your answers in a complete and clear way. You may use the back of the pages or the additional pages at the end if you need more space for your work.
- (6) You must answer all the questions.

1. [7 points] Solve the Initial Value Problem:

$$\underbrace{(6xy^2 - 3x^2y^3 + 2y)}_M dx + \underbrace{(9x^2y - 4x^3y^2 + 4x)}_N dy = 0, \quad y(1) = 1.$$

Solution $M_y = 12xy - 9x^2y^2 + 2$, $N_x = 18xy - 12x^2y^2 + 4$; NOT Exact

$$\frac{M_y - N_x}{M} = \frac{-6xy + 3x^2y^2 - 2}{-y(-6xy + 3x^2y^2 - 2)} = -\frac{1}{y} = g(y).$$

The integrating factor is $\mu(y) = e^{-\int (\frac{1}{y}) dy} = e^{\ln y} = y.$

Multiply the ODE with y :

$$\underbrace{(6xy^3 - 3x^2y^4 + 2y^2)}_{M^*} dx + \underbrace{(9x^2y^2 - 4x^3y^3 + 4xy)}_{N^*} dy = 0$$

$$M_y^* = 18xy^2 - 12x^2y^3 + 4y, \quad N_x^* = 18xy^2 - 12x^2y^3 + 4y; \text{ Now Exact}$$

Look for a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = M^*$, $\frac{\partial F}{\partial y} = N^*$

$$\frac{\partial F}{\partial x} = M^* = 6xy^3 - 3x^2y^4 + 2y^2 \Rightarrow F(x, y) = 3x^2y^3 - x^3y^4 + 2xy^2 + h(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = 9x^2y^2 - 4x^3y^3 + 4xy + h'(y) = N^* \Rightarrow h'(y) = 0$$

$$\Rightarrow h(y) = k. \text{ So } F(x, y) = 3x^2y^3 - x^3y^4 + 2xy^2 + k.$$

The general solution is $3x^2y^3 - x^3y^4 + 2xy^2 = C$

Now $y(1) = 1 \Rightarrow 3 - 1 + 2 = C \Rightarrow C = 4.$

The unique solution is $\boxed{3x^2y^3 - x^3y^4 + 2xy^2 = 4}$

2. [4 points] Solve the Initial Value Problem:

$$y' - \frac{2}{x}y = 2x^3, \quad y(1) = 0.$$

Solution This is a first-order linear ODE with

$f(x) = -\frac{2}{x}$, $r(x) = 2x^3$. The general solution is

$$y(x) = \frac{\int e^{\int -\frac{2}{x} dx} \cdot 2x^3 dx + C}{e^{\int -\frac{2}{x} dx}} = \frac{\int e^{-2 \ln x} 2x^3 dx + C}{e^{-2 \ln x}}$$

$$= \frac{\int \frac{1}{x^2} \cdot 2x^3 dx + C}{\frac{1}{x^2}} = x^2 \left[2 \int x dx + C \right] =$$

$$x^2(x^2 + C) = x^4 + Cx^2$$

Now $y(1) = 0 \Rightarrow 1 + C = 0 \Rightarrow C = -1$

So the unique solution is $\boxed{y(x) = x^4 - x^2}$

3. [4 points] Give the general solution for each of the following ODEs:

(1) $y'' - 6y' + 9y = 0$

(2) $y'' - 4y' + 5y = 0$

Solution (1) The characteristic equation is

$$\lambda^2 - 6\lambda + 9 = 0 \Leftrightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3 \text{ a double}$$

real root. The general solution is $\boxed{y(x) = C_1 e^{3x} + C_2 x e^{3x}}$

(2) The characteristic equation is $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow$

$$\lambda_1 = \frac{4 + \sqrt{16 - 4(1)(5)}}{2} = \frac{4 + \sqrt{-4}}{2} = \frac{4 + 2i}{2} = 2 + i \text{ and}$$

$$\lambda_2 = 2 - i \quad : \text{ 2 complex conjugate roots.}$$

The general solution is

$$\boxed{y(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x}$$

4. [5 points] Solve the Initial Value Problem:

$$x^2 y'' + xy' + 9y = 0, \quad x > 0, \quad y(1) = 1, \quad y'(1) = 3$$

Solution This is an Euler-Cauchy Equation with $a=1$, $b=9$. The characteristic equation is

$$m^2 + (a-1)m + b = 0 \Leftrightarrow m^2 + 9 = 0 \Rightarrow m_1 = 3i, m_2 = -3i$$

2 complex conjugate roots. The general solution is

$$y(x) = C_1 \cos(3 \ln x) + C_2 \sin(3 \ln x)$$

$$\text{Now } y'(x) = -\frac{3}{x} C_1 \sin(3 \ln x) + \frac{3}{x} C_2 \cos(3 \ln x)$$

$$y(1) = 1 \Rightarrow C_1 = 1$$

$$y'(1) = 3 \Rightarrow 3C_2 = 3 \Rightarrow C_2 = 1$$

The unique solution is $y(x) = \cos(3 \ln x) + \sin(3 \ln x)$

5. [4 points] Give the general solution for the following ODE:

$$y''' + 3y'' - y' - 3y = 0.$$

Solution The characteristic equation is

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0 \Rightarrow \lambda^2(\lambda + 3) - (\lambda + 3) = 0 \Leftrightarrow$$

$$(\lambda + 3)(\lambda^2 - 1) = 0 \Rightarrow \lambda_1 = -3, \lambda_2 = -1, \lambda_3 = 1$$

3 distinct real roots.

The general solution is $y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^x$

6. [6 points] Use the fixed point iteration method to find the root of

$$x^4 - 7x + 3 = 0$$

in the interval $[0, 1]$ to 4 decimal places. Use $x_0 = 0.45$ and make sure that the conditions for convergence of the iteration sequence are satisfied.

Solution Let $f(x) = x^4 - 7x + 3$, then f is clearly continuous and $f(0) = 3 > 0$, $f(1) = -3 < 0$. By the intermediate value theorem, f must have a root in $[0, 1]$.

$x^4 - 7x + 3 = 0 \Leftrightarrow x = \frac{x^4 + 3}{7}$. Let $g(x) = \frac{x^4 + 3}{7}$. Clearly,

g is continuous and $g'(x) = \frac{4x^3}{7}$

$|g'(x)| = \frac{4}{7} |x^3| \leq \frac{4}{7}$ on $[0, 1]$. As $\frac{4}{7} < 1$, we

know that the iteration sequence will converge.

$$x_0 = 0.45 \Rightarrow x_1 = g(x_0) = \frac{(0.45)^4 + 3}{7} = 0.4344$$

$$x_2 = g(x_1) = \frac{(0.4344)^4 + 3}{7} = 0.4337$$

$$x_3 = g(x_2) = \frac{(0.4337)^4 + 3}{7} = 0.4336$$

$$x_4 = g(x_3) = \frac{(0.4336)^4 + 3}{7} = 0.4336 \quad \text{stop.}$$

So $x \approx 0.4336$ to 4 decimal places