MAT 2384 A
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
MIDTERM

October 24, 2006

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the exam has all 6 pages.
- There are 4 questions worth a total of 30 marks.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

Question 1. (7 marks) Solve the initial value problem: A

$$(4xy + 5y^2)\,dx + (6x^2 + 20xy - 21)\,dy = 0, \quad y(1) = 1.$$

$M(x,y) = 4xy + 5y^2 \Rightarrow M_y = 4x + 10y$

$N(x,y) = 6x^2 + 20xy - 21 \Rightarrow N_x = 12x + 20y$

$\left.\right\}$ $M_y \neq N_x$ so the DE is not exact

$M_y - N_x = -8x - 10y$ so $\dfrac{M_y - N_x}{M} = \dfrac{-8x - 10y}{4xy + 5y^2} = -\dfrac{2}{y}$ (function of y only)

the integrating factor is $\mu(y) = e^{-\int -\frac{2}{y}dy} = e^{2\ln y} = y^2$

and the DE becomes $(4xy^3 + 5y^4)dx + (6x^2y^2 + 20xy^3 - 21y^2)dy = 0$

then $M^*(x,y) = 4xy^3 + 5y^4 \Rightarrow M^*_y = 12xy^2 + 20y^3$

$N^*(x,y) = 6x^2y^2 + 20xy^3 - 21y^2 \Rightarrow N^*_x = 12xy^2 + 20y^3$

$\left.\right\}$ $M^*_y = N^*_x$ so the DE is now exact

$F(x,y) = \int N^*(x,y)\,dy + g(x) = \int (6x^2y^2 + 20xy^3 - 21y^2)\,dy + g(x)$

$= 2x^2y^3 + 5xy^4 - 7y^3 + g(x)$

then $\dfrac{\partial F}{\partial x} = 4xy^3 + 5y^4 + g'(x) = M^*(x,y) = 4xy^3 + 5y^4$

so $g'(x) = 0 \Rightarrow g(x) = \text{constant}$, we take $g(x) = 0$

then $F(x,y) = 2x^2y^3 + 5xy^4 - 7y^3$

and the general solution is $2x^2y^3 + 5xy^4 - 7y^3 = C$

$y(1) = 1 \Rightarrow 2(1)^2(1)^3 + 5(1)(1)^4 - 7(1)^3 = C \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{2x^2y^3 + 5xy^4 - 7y^3 = 0}$

Question 1. (7 marks) Solve the initial value problem:

B

$$(3y + 8xy^2)\,dx + (9x + 16x^2y - 18)\,dy = 0, \qquad y(1) = 1\,.$$

$M_y = 3 + 16xy \qquad N_x = 9 + 32xy \qquad$ not exact

$M_y - N_x = -6 - 16xy\,, \qquad \dfrac{M_y - N_x}{M} = \dfrac{-6 - 16xy}{3 + 8xy^2} = -\dfrac{2}{y}$

So $\mu(y) = y^2$ and DE becomes

$$(3y^3 + 8xy^4)\,dx + (9xy^2 + 16x^2y^3 - 18y^2)\,dy = 0$$

$M^*_y = 9y^2 + 32xy^3 \qquad N^*_x = 9y^2 + 32xy^3 \qquad$ DE now exact

$$F(x,y) = \int N^*(x,y)\,dy + g(x) = 3xy^3 + 4x^2y^4 - 6y^3 + g(x)$$

$g(x) = 0$

So $F(x,y) = 3xy^3 + 4x^2y^4 - 6y^3$

general solution is $\quad 3xy^3 + 4x^2y^4 - 6y^3 = C$

$y(1) = 1 \quad \Rightarrow \quad 3(1)(1)^3 + 4(1)^2(1)^4 - 6(1)^3 = C \quad \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{3xy^3 + 4x^2y^4 - 6y^3 = 1}$

Question 2. (9 marks) Solve the initial value problems: A

(a) $y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 0$

the characteristic equation is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \quad \lambda_1 = \lambda_2 = -2$

the general solution is $y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$

$y(0) = 2 \Rightarrow 2 = C_1 e^0 + C_2 (0) e^0 \Rightarrow C_1 = 2$

$y'(x) = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x}$

$y'(0) = 0 \Rightarrow 0 = -2C_1 e^0 + C_2 e^0 - 2C_2(0) e^0 \Rightarrow -2C_1 + C_2 = 0$

$\Rightarrow C_2 = 4$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^{-2x} + 4xe^{-2x}}$

(b) $y'' - 4y' + 8y = 0, \quad y(0) = 3, \quad y'(0) = 6$

char. eq. $\lambda^2 - 4\lambda + 8 = 0 \quad \lambda_{1,2} = \dfrac{4 \pm \sqrt{(-4)^2 - 4(8)}}{2} = 2 \pm 2i$

general solution $y(x) = C_1 e^{2x} \cos(2x) + C_2 e^{2x} \sin(2x)$

$y(0) = 3 \Rightarrow 3 = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) \Rightarrow C_1 = 3$

$y'(x) = 2C_1 e^{2x} \cos(2x) - 2C_1 e^{2x} \sin(2x) + 2C_2 e^{2x} \sin(2x) + 2C_2 e^{2x} \cos(2x)$

$y'(0) = 6 \Rightarrow 6 = 2C_1 e^0 \cos(0) - 2C_1 e^0 \sin(0) + 2C_2 e^0 \sin(0) + 2C_2 e^0 \cos(0)$

$\Rightarrow 2C_1 + 2C_2 = 6 \Rightarrow C_2 = 0$

$\therefore$ the unique solution is $\boxed{y(x) = 3e^{2x} \cos(2x)}$

A

(c) $x^2y'' + xy' + 9y = 0, \quad x > 0, \quad y(1) = 1, \quad y'(1) = -2$

char. eq. $m(m-1) + m + 9 = m^2 + 9 = 0 \Rightarrow m_{1,2} = \pm 3i$

the general solution is $y(x) = C_1 \cos(3\ln x) + C_2 \sin(3\ln x)$

$y(1) = 1 \Rightarrow \quad 1 = C_1 \cos(0) + C_2 \sin(0) \Rightarrow \quad C_1 = 1$

$y'(x) = -\frac{3}{x} C_1 \sin(3\ln x) + \frac{3}{x} C_2 \cos(3\ln x)$

$y'(1) = -2 \Rightarrow -2 = -\frac{3}{1} C_1 \sin(0) + \frac{3}{1} C_2 \cos(0) \Rightarrow \quad 3C_2 = -2$

so $C_2 = -2/3$

$\therefore$ the unique solution is $\boxed{y(x) = \cos(3\ln x) - \frac{2}{3}\sin(3\ln x)}$

Question 2. (9 marks) Solve the initial value problems: B

(a) $y'' - 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 0$

char. eq. $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2$

general solution $y(x) = c_1 e^{2x} + c_2 x e^{2x}$

$y(0) = 2 \Rightarrow 2 = c_1 e^0 + c_2 (0) e^0 \Rightarrow c_1 = 2$

$y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$

$y'(0) = 0 \Rightarrow 0 = 2c_1 e^0 + c_2 e^0 + 2c_2(0)e^0 \Rightarrow 2c_1 + c_2 = 0$
so $c_2 = -4$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^{2x} - 4xe^{2x}}$

(b) $y'' + 4y' + 8y = 0, \quad y(0) = 3, \quad y'(0) = -6$

char. eq. $\lambda^2 + 4\lambda + 8 = 0 \qquad \lambda_{1,2} = \dfrac{-4 \pm \sqrt{(4)^2 - 4(8)}}{2} = -2 \pm 2i$

general solution $y(x) = c_1 e^{-2x} \cos(2x) + c_2 e^{-2x} \sin(2x)$

$y(0) = 3 \Rightarrow 3 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) \Rightarrow c_1 = 3$

$y'(x) = -2c_1 e^{-2x} \cos(2x) - 2c_1 e^{-2x} \sin(2x) - 2c_2 e^{-2x} \sin(2x) + 2c_2 e^{-2x} \cos(2x)$

$y'(0) = -6 \Rightarrow -6 = -2c_1 e^0 \cos(0) - 2c_1 e^0 \sin(0) - 2c_2 e^0 \sin(0) + 2c_2 e^0 \cos(0)$
$\Rightarrow -2c_1 + 2c_2 = -6 \Rightarrow c_2 = 0$

$\therefore$ the unique solution is $\boxed{y(x) = 3e^{-2x} \cos(2x)}$

B

(c) $x^2y'' + xy' + 16y = 0, \quad x > 0, \quad y(1) = 1, \quad y'(1) = -3$

char. eq. $m^2 + 16 = 0 \Rightarrow m_{1,2} = \pm 4i$

general solution $y(x) = c_1 \cos(4 \ln x) + c_2 \sin(4 \ln x)$

$y(1) = 1 \Rightarrow 1 = c_1 \cos(0) + c_2 \sin(0) \Rightarrow c_1 = 1$

$y'(x) = \frac{-4}{x} c_1 \sin(4 \ln x) + \frac{4}{x} c_2 \cos(4 \ln x)$

$y'(1) = -3 \Rightarrow -3 = -\frac{4}{1} c_1 \sin(0) + \frac{4}{1} c_2 \cos(0)$

$\Rightarrow 4c_2 = -3$ so $c_2 = -\frac{3}{4}$

$\therefore$ the unique solution is $\boxed{y(x) = \cos(4 \ln x) - \frac{3}{4} \sin(4 \ln x)}$

Question 3. (8 marks) Solve the initial value problem:

A

$$y''' - 2y'' - 11y' + 12y = 0, \quad y(0) = 2, \quad y'(0) = 26, \quad y''(0) = 38.$$

the characteristic equation is $\lambda^3 - 2\lambda^2 - 11\lambda + 12 = 0$

by inspection, we can see that $\lambda = 1$ is a root, so

$\lambda^3 - 2\lambda^2 - 11\lambda + 12 = (\lambda - 1)(\lambda^2 - \lambda - 12) = (\lambda - 1)(\lambda - 4)(\lambda + 3)$

so the roots are $\lambda_1 = 1$, $\lambda_2 = 4$ and $\lambda_3 = -3$

the general solution is $y(x) = C_1 e^{x} + C_2 e^{4x} + C_3 e^{-3x}$

$y(0) = 2 \Rightarrow 2 = C_1 e^0 + C_2 e^0 + C_3 e^0 \Rightarrow C_1 + C_2 + C_3 = 2$ ①

$y'(x) = C_1 e^{x} + 4C_2 e^{4x} - 3C_3 e^{-3x}$

$y'(0) = 26 \Rightarrow 26 = C_1 e^0 + 4C_2 e^0 - 3C_3 e^0 \Rightarrow C_1 + 4C_2 - 3C_3 = 26$ ②

$y''(x) = C_1 e^{x} + 16 C_2 e^{4x} + 9 C_3 e^{-3x}$

$y''(0) = 38 \Rightarrow 38 = C_1 e^0 + 16 C_2 e^0 + 9 C_3 e^0 \Rightarrow C_1 + 16 C_2 + 9 C_3 = 38$ ③

② − ① $\quad 3C_2 - 4C_3 = 24$ ④

③ − ① $\quad 15 C_2 + 8 C_3 = 36$ ⑤

④ × 2 $\quad 6 C_2 - 8 C_3 = 48$ ⑥

⑤ + ⑥ $\quad 21 C_2 = 84 \Rightarrow C_2 = 4 \Rightarrow C_3 = -3 \Rightarrow C_1 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = e^{x} + 4e^{4x} - 3e^{-3x}}$

B

Question 3. (8 marks) Solve the initial value problem:

$$y''' - 6y'' + 5y' + 12y = 0, \quad y(0) = 6, \quad y'(0) = 26, \quad y''(0) = 90.$$

char. eq. $\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$

$\lambda = -1$ is root $\quad \lambda^3 - 6\lambda^2 + 5\lambda + 12 = (\lambda+1)(\lambda^2 - 7\lambda + 12) = (\lambda+1)(\lambda-3)(\lambda-4)$

$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \lambda_3 = 4$

general solution $\quad y(x) = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{4x}$

$y(0) = 6 \quad \Rightarrow \quad c_1 + c_2 + c_3 = 6 \quad (1)$

$y'(x) = -c_1 e^{-x} + 3c_2 e^{3x} + 4c_3 e^{4x}$

$y'(0) = 26 \quad \Rightarrow \quad -c_1 + 3c_2 + 4c_3 = 26 \quad (2)$

$y''(x) = c_1 e^{-x} + 9c_2 e^{3x} + 16c_3 e^{4x}$

$y''(0) = 90 \quad \Rightarrow \quad c_1 + 9c_2 + 16c_3 = 90 \quad (3)$

(2) + (1) $\quad 4c_2 + 5c_3 = 32 \quad (4)$

(3) − (1) $\quad 8c_2 + 15c_3 = 84 \quad (5)$

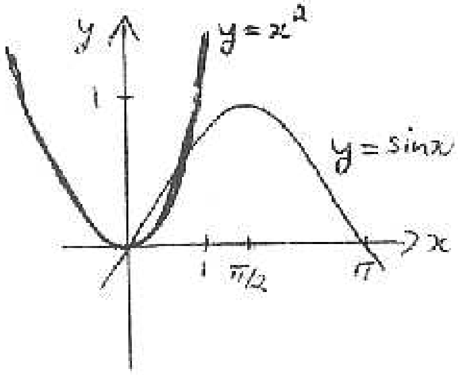
(4) × 2 $\quad 8c_2 + 10c_3 = 64 \quad (6)$

(5) − (6) $\quad 5c_3 = 20 \Rightarrow c_3 = 4 \Rightarrow c_2 = 3 \Rightarrow c_1 = -1$

$\therefore$ the unique solution is $\boxed{y(x) = 3e^{3x} + 4e^{4x} - e^{-x}}$

Question 4. (6 marks) Use Newton's Method to find the positive solution of $x^2 = \sin x$ to 6 decimal places. Start with $x_0 = 1$. Verify your answer.

A



Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $f(x) = x^2 - \sin x$, then $f'(x) = 2x - \cos x$

So $x_{n+1} = x_n - \dfrac{x_n^2 - \sin x_n}{2x_n - \cos x_n} = \dfrac{x_n^2 - x_n \cos x_n + \sin x_n}{2x_n - \cos x_n}$

(radians !)

$x_0 = 1$, $x_1 = \dfrac{(1)^2 - (1)\cos(1) + \sin(1)}{2(1) - \cos(1)} = 0.891396$

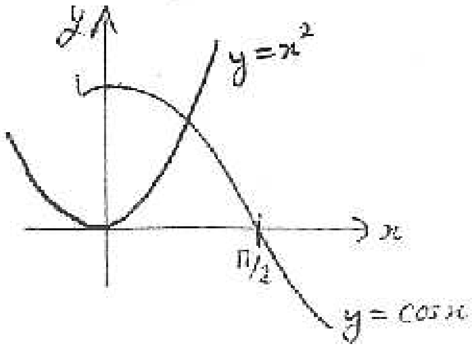
$x_2 = 0.876985$

$x_3 = 0.876726 = x_4 \quad \therefore$ stop

check: $(0.876726)^2 = 0.768648$
$\sin(0.876726) = 0.768649$ okay

$\therefore$ the solution is $\boxed{0.876726}$

Question 4. (6 marks) Use Newton's Method to find the positive solution of $x^2 - \cos x$ to 6 decimal places. Start with $x_0 = 1$. Verify your answer. B



let $f(x) = x^2 - \cos x$

then $f'(x) = 2x + \sin x$

so $$x_{n+1} = x_n - \frac{x_n^2 - \cos x_n}{2x_n + \sin x_n} = \frac{x_n^2 + x_n \sin x_n + \cos x_n}{2x_n + \sin x_n}$$

$x_0 = 1$ $\quad x_1 = 0.838218$

$x_2 = 0.824242$

$x_3 = 0.824132 = x_4$ stop

check : $$\left.\begin{array}{l}(0.824132)^2 = 0.679194 \\ \cos(0.824132) = 0.679194\end{array}\right] \text{okay}$$

$\therefore$ the solution is $\boxed{0.824132}$