4. Suppose $e, f \in \mathbb{R}$ and consider the linear system in x, y and z:

$$\begin{aligned} 2x - 2y + ez &= f \\ x + z &= -1 \\ 3x + y + 2z &= -1 \end{aligned}$$

(a) If $[\,A \,|\, b\,]$ is the augmented matrix of the system above, find rank(A) and rank$[\,A \,|\, b\,]$ for all values of e and f.

$$\left[\begin{array}{ccc|c} 2 & -2 & e & f \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \\ 2 & -2 & e & f \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & e-2 & f+2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & e-4 & f+6 \end{array}\right]$$

$R_1 \leftrightarrow R_2$, $R_3 \leftrightarrow R_2$

$-3R_1 + R_2 \to R_2$, $-2R_1 + R_3 \to R_3$

$2R_2 + R_3 \to R_3$

$$\text{Rank } A = \begin{cases} 2 & \text{when } e = 4 \\ 3 & \text{when } e \neq 4 \end{cases}$$

$$\text{Rank } [A|b] = \begin{cases} 2 & \text{when } e = 4 \text{ and } f = -6 \\ 3 & \text{when } e \neq 4 \text{ or } f \neq -6 \end{cases}$$

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(Q4 parts (b) and (c) are on the next page...)