ACTU 256 / Fall 2015

Assignment 1-Solutions

Due: Friday, September 25, 2015 (at the beginning of the class)

1. A bank started a new savings account on October 15, 2012, using the accumulation function $a(t) = p \cdot t^2 + q \cdot t + r$, where t is measured in years. An investment of 400 made on October 15, 2012, is worth 802 on October 15, 2013, and 1208 on October 15, 2014 .
(a) Find $a(t)$.
(b) Find the accumulated value of an investment of $1,500$ made on October 15, 2012, after three years.
(c) For this account (started on October 15, 2012), find the effective rate of interest during the third year.
(d) For this account (started on October 15, 2012), find the effective rate of discount during the fourth year.

Solution.
(a) October 15, 2012 (the savings account started at this time) means time $t = 0$. Therefore, October 15, 2013 means time $t = 1$ and October 15, 2014 means time $t = 2$.
An investment of 400 made on October 15, 2012 is worth 802 on October 15, 2013:

$$A(1) = A(0) \cdot a(1) \;\Rightarrow\; 802 = 400 \cdot (p + q + r) \qquad (1)$$

and is worth 1208 on October 15, 2014:

$$A(2) = A(0) \cdot a(2) \;\Rightarrow\; 1208 = 400 \cdot (4p + 2q + r) \qquad (2)$$

The accumulation function $a(t)$ satisfies:

$$a(0) = 1 \;\Rightarrow\; r = 1.$$

Using this value of r in relations (1) and (2) and solving the corresponding system of equations for p and q, it follows $p = 0.005$ and $q = 1$. So, $a(t) = 0.005t^2 + t + 1$.
(b). The accumulated value of an investment of $1,500$ made on October 15, 2012 after three years is:

$$A(3) = A(0) \cdot a(3) \;\Rightarrow\; A(3) = 1500 \cdot (0.005 \cdot 9 + 3 + 1) \;\Rightarrow\; A(3) = 6067.50.$$

(c). The effective rate of interest during the third year is

$$i_3 = \frac{a(3) - a(2)}{a(2)} = \frac{(0.005 \cdot 9 + 3 + 1) - (0.005 \cdot 4 + 2 + 1)}{0.005 \cdot 4 + 2 + 1} = 0.3394 = 33.94\%.$$

(d). The effective rate of discount during the fourth year is:

$$d_4 = \frac{a(4) - a(3)}{a(4)} = \frac{(0.005 \cdot 16 + 4 + 1) - (0.005 \cdot 9 + 3 + 1)}{0.005 \cdot 16 + 4 + 1} = 0.2037 = 20.37\%.$$

2. Robert deposits 15,000 into a bank account that pays an effective annual interest rate of 2% for ten years, with interest credited at the end of each year. At the beginning of the second year he makes a second deposit 3% greater than the previous year's deposit. If a withdrawal is made during the first 5.5 years, Robert pays a penalty of 5% of the withdrawal amount. Robert withdraws X at the beginning of the sixth year and $2X$ at the end of each of years 7 and 8. What is the amount X if Robert's bank account balance is 15,000 at the end of ten years?

Solution. At time $t = 0$ there is a deposit of 15000. At time $t = 1$ (the beginning of the second year) there is a second deposit of $15000 \cdot (1.03) = 15450$. At time $t = 5$, Robert withdraws X and hence, he pays a penalty of $0.05X$. Robert also withdraws $2X$ at time $t = 7$ and at time $t = 8$. The balance is 15,000 at the end of ten years:

$$15000 = 15000(1.02)^{10} + 15450(1.02)^9 - X(1.05)(1.02)^5 - 2X(1.02)^3 - 2X(1.02)^2$$

Solving the above equation for X, it follows that $X = 4055.77 \simeq 4056$.

3. The parents of three children aged 1, 3, and 6 wish to set up a trust fund that will pay 15,000 to each child upon attainment of age 18, and 75,000 to each child upon attainment of age 21.
(a) If the trust fund will earn a 6-month interest rate of 4% compounded every 6 months, what amount must the parents invest now in the trust fund?

(b) Assume that the trust fund will grow at nominal interest rate of 3% convertible monthly for the first four years, force of interest $\delta_t = \frac{2}{1+0.5t}$ for the next year, nominal rate of discount of 4% compounded quarterly for the next seven years and effective annual rate of discount at 5.75% thereafter.
What amount must the parents now invest in the trust fund in order to pay 25,000 to the child aged 6 upon attainment of age 18, and 100,000 to the child aged 6 upon attainment of age 21.

Solution. (a) The required amount now, denoted by A, is the sum of the present values at time zero of the payments of 15,000 to each child upon attainment of age 18, and 75,000 to each child upon attainment of age 21:

$$A = 15000 \cdot \left[\frac{1}{\left(1 + \frac{i^{(2)}}{2}\right)^{34}} + \frac{1}{\left(1 + \frac{i^{(2)}}{2}\right)^{30}} + \frac{1}{\left(1 + \frac{i^{(2)}}{2}\right)^{24}} \right]$$

$$+75000\left[\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{40}}+\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{36}}+\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{30}}\right]=71450.62,$$

where $\frac{i^{(2)}}{2}=0.04$.

(b) For the child aged 6, the required amount now is:

$$25000\cdot\frac{1}{\left(1+\frac{0.03}{12}\right)^{48}}\cdot e^{-\int_4^5\frac{2}{1+0.5t}dt}\cdot\left(1-\frac{0.04}{4}\right)^{28}$$

$$+100000\cdot\frac{1}{\left(1+\frac{0.03}{12}\right)^{48}}\cdot e^{-\int_4^5\frac{2}{1+0.5t}dt}\cdot\left(1-\frac{0.04}{4}\right)^{28}\cdot(1-0.0575)^3\approx 39289.$$

where

$$\int_4^5\frac{2}{1+0.5t}dt=4\ln(1+0.5t)|_4^5=0.6166.$$

4. (a) Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of 12% per year convertible quarterly.

(b) Find the nominal rate of interest convertible daily which is equivalent to a nominal rate of discount of 6% per year convertible daily. (Assume a non-leap year).

(c) Establish the following relationships:

$$\frac{i^{(m)}}{m}=\frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}};\quad\frac{d^{(m)}}{m}=\frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}};\quad\frac{i^{(m)}}{m}-\frac{d^{(m)}}{m}=\frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m};\quad i^{(m)}=d^{(m)}(1+i)^{\frac{1}{m}},$$

where m is a positive integer.

(d) If $i^{(m)}$ is the nominal annual interest rate and i is the equivalent simple interest rate per year, what is the relationship between i and $i^{(m)}$. Similarly, if $d^{(m)}$ is the nominal annual discount rate and d is the equivalent simple discount rate per year, what is the relationship between d and $d^{(m)}$. In both cases, m is a positive integer and a simple model is assumed.

Solution. (a) We have

$$\left(1-\frac{d^{(2)}}{2}\right)^{-2}=\left(1+\frac{i^{(4)}}{4}\right)^4$$

$$\Rightarrow\ d^{(2)}=2\cdot\left[1-\left(1+\frac{i^{(4)}}{4}\right)^{-2}\right]=2\cdot\left[1-\left(1+\frac{0.12}{4}\right)^{-2}\right]=0.1148=11.48\%.$$

(b) Assuming 365 days in a year, we have

$$\left(1+\frac{i^{(365)}}{365}\right)^{365} = \left(1-\frac{d^{(365)}}{365}\right)^{-365}$$

$$\Rightarrow\ i^{(365)} = 365\cdot\left[\left(1-\frac{d^{(365)}}{365}\right)^{-1} - 1\right] = 365\cdot\left[\left(1-\frac{0.06}{365}\right)^{-1} - 1\right] = 0.06 = 6\%.$$

(c) The relationship

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \left(1-\frac{d^{(m)}}{m}\right)^{-m}$$

leads to

$$1+\frac{i^{(m)}}{m} = \left(1-\frac{d^{(m)}}{m}\right)^{-1} \Leftrightarrow \left(1+\frac{i^{(m)}}{m}\right)\cdot\left(1-\frac{d^{(m)}}{m}\right) = 1$$

$$\Leftrightarrow \frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} - \frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m} = 0. \qquad (2)$$

Solving (2) for $\frac{i^{(m)}}{m}$, it follows:

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}}.$$

Solving (2) for $\frac{d^{(m)}}{m}$, it follows:

$$\frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}.$$

Rearranging terms in (2) yields

$$\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m}.$$

By using the following two relations:

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}} \quad \text{and} \quad \left(1-\frac{d^{(m)}}{m}\right)^{-m} = 1+i,$$

the desired result is obtained, that is,

$$i^{(m)} = d^{(m)}(1+i)^{\frac{1}{m}}.$$

(d) Under simple interest model, if $i^{(m)}$ and i are equivalent, then the accumulated value at the end of one year is:

$$1 + \frac{i^{(m)}}{m} \cdot m = 1 + i \quad \Rightarrow \quad i^{(m)} = i.$$

Under simple discount model, if $d^{(m)}$ and d are equivalent, then the present value at time zero is:

$$1 - \frac{d^{(m)}}{m} \cdot m = 1 - d \quad \Rightarrow \quad d^{(m)} = d.$$

5. On January 1, 2011, Smith deposits $1,000$ into an account earning nominal annual interest rate of $i^{(4)} = 0.04$ compounded quarterly with interest credited on the last day of March, June, September, and December. If Smith closes the account during the year, simple interest of 4% is paid on the balance from the most recent interest credit date.
(a) What is Smith's close-out balance on August 20, 2011?

(b) Suppose all four quarters in the year are considered equal, and time is measured in years. Derive expressions for Smith's accumulated amount function $A(t)$, the close-out balance at time t. Consider separately the four intervals $0 \leq t \leq 0.25$, $0.25 \leq t \leq 0.50$, $0.50 \leq t \leq 0.75$ and $0.75 \leq t \leq 1$ and draw the time diagram for each of these cases.

(c) Using part (b), show that if $0 \leq t \leq 0.25$, then it follows that $\delta_t = \delta_{t+0.25} = \delta_{t+0.50} = \delta_{t+0.75}$.
Solution. If Smith closes the account during the year, simple interest is paid on *the balance from the most recent interest credit date.*
Since $i^{(4)} = 0.04$, then $\frac{i^{(4)}}{4} = 0.01$ is $\frac{1}{4}$-year effective interest rate. Under simple interest model, we have that $i^{(4)} = i = 0.04$.
(a) Smith's close-out balance on August 20, 2011 is:

$$1000 \cdot \left(1 + \frac{0.04}{4}\right)^2 \cdot \left(1 + \frac{51}{365} \cdot 0.04\right) = 1025.8013,$$

since Smith closes the account during the third quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^2$.
(b) $A(t)$ is the amount function at time t.

- If $0 \leq t \leq 0.25$, then $A(t) = 1000 \cdot (1 + 0.04 \cdot t)$, since Smith closes the account during the first quarter and simple interest is paid on the balance of 1000.

- If $0.25 \le t \le 0.50$, then $A(t) = 1000 \cdot \left(1 + \frac{0.04}{4}\right)^1 \cdot (1 + 0.04 \cdot (t - 0.25))$, since Smith closes the account during the second quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^1$.
- If $0.50 \le t \le 0.75$, then $A(t) = 1000 \cdot \left(1 + \frac{0.04}{4}\right)^2 \cdot (1 + 0.04 \cdot (t - 0.50))$, since Smith closes the account during the third quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^2$.
- If $0.75 \le t \le 1$, then $A(t) = 1000 \cdot \left(1 + \frac{0.04}{4}\right)^3 \cdot (1 + 0.04 \cdot (t - 0.75))$, since Smith closes the account during the fourth quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^3$.

Consequently,

$$A(t) = \begin{cases} 1000 \cdot (1 + 0.04 \cdot t) & \text{for } 0 \le t \le 0.25 \\ 1000 \cdot (1.01) \cdot (1 + 0.04 \cdot (t - 0.25)) & \text{for } 0.25 \le t \le 0.50 \\ 1000 \cdot (1.01)^2 \cdot (1 + 0.04 \cdot (t - 0.50)) & \text{for } 0.50 \le t \le 0.75 \\ 1000 \cdot (1.01)^3 \cdot (1 + 0.04 \cdot (t - 0.75)) & \text{for } 0.75 \le t \le 1. \end{cases}$$

(c) For $0 \le t \le 0.25$, we have:

- $\delta_t = \frac{A'(t)}{A(t)} = \frac{[1000 \cdot (1+0.04 \cdot t)]'}{1000 \cdot (1+0.04 \cdot t)} = \frac{0.04}{1+0.04 \cdot t}$.
- $0.25 \le t + 0.25 \le 0.50$, and if we denote $t_1 = t + 0.25$, we have
$$\delta_{t_1} = \frac{A'(t_1)}{A(t_1)} = \frac{[1000 \cdot (1.01) \cdot (1 + 0.04 \cdot (t_1 - 0.25))]'}{1000 \cdot (1.01) \cdot (1 + 0.04 \cdot (t_1 - 0.25))} = \frac{0.04}{1 + 0.04 \cdot (t_1 - 0.25)}$$
and hence,
$$\delta_{t+0.25} = \frac{0.04}{1 + 0.04 \cdot (t + 0.25 - 0.25)} = \frac{0.04}{1 + 0.04 \cdot t} = \delta_t.$$
- $0.50 \le t + 0.50 \le 0.75$, and if we denote $t_2 = t + 0.50$, we have
$$\delta_{t_2} = \frac{A'(t_2)}{A(t_2)} = \frac{[1000 \cdot (1.01)^2 \cdot (1 + 0.04 \cdot (t_2 - 0.50))]'}{1000 \cdot (1.01)^2 \cdot (1 + 0.04 \cdot (t_2 - 0.50))} = \frac{0.04}{1 + 0.04 \cdot (t_2 - 0.50)}$$
and hence,
$$\delta_{t+0.50} = \frac{0.04}{1 + 0.04 \cdot (t + 0.50 - 0.50)} = \frac{0.04}{1 + 0.04 \cdot t} = \delta_t.$$
- $0.75 \le t + 0.75 \le 1$, and if we denote $t_3 = t + 0.75$, we have
$$\delta_{t_3} = \frac{A'(t_3)}{A(t_3)} = \frac{[1000 \cdot (1.01)^3 \cdot (1 + 0.04 \cdot (t_3 - 0.75))]'}{1000 \cdot (1.01)^3 \cdot (1 + 0.04 \cdot (t_3 - 0.75))} = \frac{0.04}{1 + 0.04 \cdot (t_3 - 0.75)}$$
and hence,
$$\delta_{t+0.75} = \frac{0.04}{1 + 0.04 \cdot (t + 0.75 - 0.75)} = \frac{0.04}{1 + 0.04 \cdot t} = \delta_t.$$