Final test - December 2015

1.(10) (a) $f(x) = \begin{cases} 1+\sqrt{4-x^2} & if \quad -2 \le x \le 2 \\ 3-x & if \quad x > 2 \end{cases}$ → draw the graph on $[-2,5]$

and use it to evaluate: $\int\limits_{-2}^{5} f(x)\,dx$.

(b) Calculate $F'(x)$ if $F(x) = \int\limits_{-x^2}^{x} e^{1-t^2}dt$ and determine whether $F(x)$ is increasing, or decreasing.

2.(10) Evaluate:

(a) $\int \dfrac{\cos^3 x}{\sin^2 x}dx$

(b) $\int (e^x + \ln x)\,dx$

3.(6) Calculate $F(x)$ if $F'(x) = \dfrac{x^2+4}{x^2-4}$ and $F(-1) = 0$.

4.(18) Evaluate:

(a) $\int\limits_{0}^{\pi/2} \dfrac{\cos x}{4+\sin^2 x}dx$

(b) $\int\limits_{0}^{\pi/4} \sec^4 x dx$

(c) $\int\limits_{0}^{3} x^2\sqrt{1+x}dx$

5.(8) Evaluate the improper integrals, or show why they are divergent:

(a) $\int\limits_{e}^{\infty} \dfrac{dx}{x\ln^3 x}$

(b) $\int\limits_{-1}^{1} \dfrac{1dx}{x^2-1}$

6.(16) (a) Sketch and calculate the area enclosed by $y = 6 - x^2$ & $y = 2 - 3x$.

(b) Evaluate the volume of a solid obtained by rotating the region bounded by $y = \sin x$ and the $x - axis$ on $[0, \pi]$ about $y = -1$.

(c) Calculate f_{ave} of $f(x) = \dfrac{x}{\sqrt{16 + x^2}}$ on $[0, 3]$.

7.(6) Calculate $\lim a_n$ or prove that it does not exist:

(a) $a_n = \dfrac{(3^n + 1)^2}{6^n}$

(b) $a_n = \ln\left(1 + 2n^2\right) - \ln\left(30 + 2n^2\right)$

8.(12) Determine whether the series $\sum_{n=2}^{\infty} a_n$ is divergent, absolutely convergent, or conditionally convergent:

(a) $a_n = \dfrac{(-1)^n \sqrt{1 + n^3}}{n^2}$

(b) $a_n = \dfrac{(-3)^n}{5 + e^n}$

(c) $a_n = \dfrac{1}{n \ln^2 n}$

9.(6) Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \dfrac{(x+1)^n}{(n+1)\, 2^n}$.

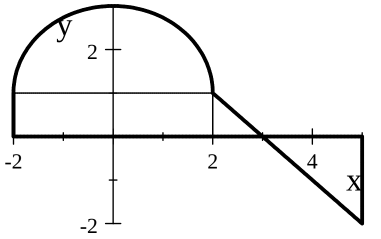
10(8) (a) Calculate the MacLaurin series for $f(x) = x^2 \ln(1 + 2x)$

(b) Calculate $S(x) = \sum_{n=1}^{\infty} n x^{2n-1}$ and determine its radius of convergence.

Bonus(5) Show that for a continuous f : $\int_0^{\pi/2} f(\cos x)\, dx = \int_0^{\pi/2} f(\sin x)\, dx$.

Solutions:

1. (a) $f(x) = \begin{cases} 1+\sqrt{4-x^2} & if \quad -2 \le x \le 2 \\ 3-x & if \quad x > 2 \end{cases}$ → draw the graph on $[-2,5]$



and use it to evaluate: $\int\limits_{-2}^{5} f(x)\,dx = 2\pi + 4 + .5 - 2 = 2\pi + 2.5$

(b) $F'(x) = \dfrac{d}{dx}\int\limits_{-x^2}^{x} e^{1-t^2}dt = e^{1-x^2} + 2xe^{1-x^4}$ (3)→ $F'(1) = 1+2 > 0$ → it is increasing.

2. Evaluate:

(a) $\int \dfrac{\cos^3 x}{\sin^2 x}dx = \int \dfrac{\left(1-\sin^2 x\right)\cos x dx}{\sin^2 x} = \left| \; t = \sin x \quad dt = \cos x dx \; \right| = \int \dfrac{1-t^2}{t^2}dt = C - \dfrac{1}{t}\left(t^2+1\right) = C - \dfrac{1}{\sin x}\left(\sin^2 x + 1\right) = C - \sin x - \dfrac{1}{\sin x}.$

(b) $\int \left(e^x + \ln x\right)dx = e^x + \int \ln x dx = \begin{vmatrix} u = \ln x & v' = 1 \\ u' = \dfrac{1}{x} & v = x \end{vmatrix} = e^x + x\ln x - \int dx = e^x - x + x\ln x + C$

3. $F'(x) = \dfrac{x^2+4}{x^2-4} = 1 + \dfrac{8}{(x-2)(x+2)} = 1 + \dfrac{A}{x-2} + \dfrac{B}{x+2} \rightarrow 8 = A(x+2) + B(x-2);$

If $x = -2 : 8 = -4B \rightarrow B = -2$ and if $x = 2 : 8 = 4A \rightarrow A = 2$. Therefore, $F(x) = \int \left(1 + \dfrac{2}{x-2} - \dfrac{2}{x+2}\right)dx = x + 2\ln|x-2| - 2\ln|x+2| + C$. If $F(-1) = 0$ then: $-1 + 2\ln|-1-2| - 2\ln|-1+2| + C = C + 2\ln 3 - 1 = 0 \rightarrow C = 1 - 2\ln 3 \rightarrow F(x) = x + 2\ln|x-2| - 2\ln|x+2| + 1 - 2\ln 3$.

4. Evaluate:

(a) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{4+\sin^2 x}dx = \left| \begin{array}{cc} t=\sin x & dt=\cos x dx \\ x=0 \rightarrow t=0 & x=\dfrac{\pi}{2} \rightarrow t=1 \end{array} \right| = \int_0^1 \frac{dt}{4+t^2} =$
$\displaystyle\frac{1}{2}\arctan\frac{t}{2}]_0^1 = \frac{1}{2}\arctan\frac{1}{2}$

(b) $\displaystyle\int_0^{\pi/4} \sec^4 x dx = \int_0^{\pi/4} \sec^2 x \left(1+\tan^2 x\right) dx = \left| \begin{array}{cc} t=\tan x & dt=\sec^2 x dx \\ x=0 \rightarrow t=0 & x=\dfrac{\pi}{4} \rightarrow t=1 \end{array} \right| =$
$\displaystyle\int_0^1 \left(1+t^2\right) dt = \frac{4}{3}$

(c) $\displaystyle\int_0^3 x^2\sqrt{1+x}dx = \left| \begin{array}{cc} t=1+x & dt=dx \\ x=0 \rightarrow t=1 & x=3 \rightarrow t=4 \end{array} \right| = \int_1^4 (t-1)^2\sqrt{t}dx =$
$\displaystyle\int_1^4 \left(t^{1/2}-2t^{3/2}+t^{5/2}\right) dt = \frac{1,696}{105}$

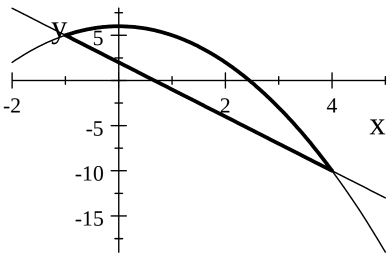
5. Evaluate the improper integrals, or show why they are divergent:

(a) $\displaystyle\int_e^{\infty} \frac{dx}{x\ln^3 x} = \left| \begin{array}{cc} t=\ln x & dt=\dfrac{dx}{x} \\ x=e \rightarrow t=1 & x\rightarrow\infty \rightarrow t\rightarrow\infty \end{array} \right| = \lim_{s\rightarrow\infty}\int_1^s \frac{dt}{t^3} = \frac{1}{2}.$

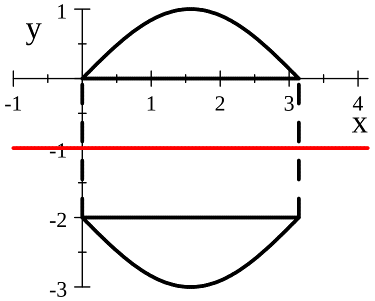
(b) $\displaystyle\int_{-1}^1 \frac{xdx}{x^2-1} = \left| \frac{x}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1} \rightarrow x = A(x+1)+B(x-1) \rightarrow A=B=\frac{1}{2} \right|$
$\displaystyle= \frac{1}{2}\left[\int_{-1}^0 \left(\frac{1}{x-1}+\frac{1}{x+1}\right)dx + \int_0^1 \left(\frac{1}{x-1}+\frac{1}{x+1}\right)dx\right] = \lim_{s\rightarrow -1^+}\frac{1}{2}\ln\left|s^2-1\right| -$
$\displaystyle\lim_{s\rightarrow 1^-}\frac{1}{2}\ln\left|s^2-1\right|$ divergent.

(a) Sketch and calculate the area enclosed by $y=6-x^2$ & $y=2-3x$.

$6-x^2=2-3x$, giving: $x=-1, x=4 \rightarrow Area = \displaystyle\int_{-1}^4 \left(\left(6-x^2\right)-(2-3x)\right)dx =$
$\dfrac{125}{6}$



(b) Evaluate the volume of a solid obtained by rotating the region bounded by $y = \sin x$ and the $x - axis$ on $[0, \pi]$ about $y = -1$.



Volume $V = \pi \int\limits_0^\pi (\sin x + 1)^2 \, dx - \pi = \pi \int\limits_0^\pi \left(\sin^2 x + 2\sin x + 1\right) dx - \pi = \pi \left(\frac{3\pi}{2} + 3\right)$.

(c) Calculate f_{ave} of $f(x) = \frac{x}{\sqrt{16 + x^2}}$ on $[0, 3]$.

$f_{ave} = \frac{1}{3} \int\limits_0^3 \frac{xdx}{\sqrt{16 + x^2}} = \left| \begin{array}{cc} t = \sqrt{16 + x^2} & tdt = xdx \\ x = 0 \to t = 4 & x = 3 \to t = 5 \end{array} \right| = \frac{1}{3} \int\limits_4^5 dt = \frac{1}{3}$.

6. Calculate $\lim a_n$ or prove that it does not exist:

(a) $a_n = \frac{(3^n + 1)^2}{6^n} \to \lim \frac{(3^n + 1)^2}{6^n} = \lim \frac{9^n + 2 \times 3^n + 1}{6^n} = \infty$ - divergent

(b) $a_n = \ln\left(1 + 2n^2\right) - \ln\left(30 + 2n^2\right) \to \lim \ln \frac{1 + 2n^2}{30 + 2n^2} = \lim \ln \frac{\frac{1}{n^2} + 2}{\frac{30}{n^2} + 2} = 0$.

7. Determine whether the series $\sum_{n=2}^{\infty} a_n$ is divergent, absolutely convergent, or conditionally convergent:

   (a) $a_n = \frac{(-1)^n \sqrt{1+n^3}}{n^2} \to \lim |a_n| = \lim \frac{\sqrt{1+n^3}}{n^2} = \lim \frac{\sqrt{\frac{1}{n^4}+\frac{1}{n}}}{1} \searrow$ 0 therefore convergent. Since $p = 2 - \frac{3}{2} = \frac{1}{2} < 1 \to$ the convergence is conditional

   (b) $a_n = \frac{(-3)^n}{5+e^n} \to \lim |a_n| = \lim \frac{3^n}{5+e^n} = \lim \frac{\left(\frac{3}{e}\right)^n}{\frac{5}{e^n}+1} = \infty$ the series is not convergent.

   (c) $a_n = \frac{1}{n\ln^2 n} \to$ Since $\int\limits_2^{\infty} \frac{dx}{x\ln^2 x} = \left| \begin{array}{cc} t = \ln x & tdt = xdx \\ x = 2 \to t = \ln 2 & x \to \infty \to t \to \infty \end{array} \right| =$ $\lim\limits_{s\to\infty} \int\limits_{\ln 2}^{s} \frac{dt}{t^2} = \frac{1}{\ln 2} - \lim\limits_{s\to\infty} \frac{1}{s} = \frac{1}{\ln 2}$ $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ is convergent (absolutely).

8. Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1)\,2^n}$.

   The radius $r = \lim \frac{a_n}{a_{n+1}} = \lim \frac{(n+2)\,2^{n+1}}{(n+1)\,2^n} = 2\lim \frac{n+2}{n+1} = 2$ and center $a = -1$ giving interval $(-3,1)$.At the end-points: $x = -3 \to$ $\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)\,2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}$ as $\frac{1}{n+1} \searrow 0$ it is convergent. At $x = 1 \to$ $\sum_{n=0}^{\infty} \frac{2^n}{(n+1)\,2^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ with $p = 1 \ngtr 1 \to$ divergent. The interval of convergence is then $[-3,1)$.

   (a) Calculate the MacLaurin series for $f(x) = x^2\ln(1+2x)$. As the MacLaurin series for $\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ on interval $(-1,1]$ we get $f(x) = x^2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2x)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} x^{n+2}$ on $(-\frac{1}{2},\frac{1}{2}]$

   (b) Calculate $S(x) = \sum_{n=1}^{\infty} nx^{2n-1}$ and determine its radius of convergence. Since $\int \sum_{n=1}^{\infty} x^{2n-1} = \sum_{n=1}^{\infty} n \int x^{2n-1} dx = \sum_{n=1}^{\infty} \frac{n}{2n} x^{2n} = \frac{1}{2}\frac{1}{1-x^2}$

if $x^2 < 1 \to x \in (-1,1)$. Therefore, $S(x) = \sum\limits_{n=1}^{\infty} n x^{2n-1} = \frac{d}{dx}\frac{1}{2}\frac{1}{1-x^2} = \frac{x}{\left(x^2-1\right)^2}$ on $(-1,1)$.

Bonus Show that for a continuous f : $\int\limits_0^{\pi/2} f(\cos x)\,dx = \int\limits_0^{\pi/2} f(\sin x)\,dx$.

$$\int\limits_0^{\pi/2} f(\cos x)\,dx = \int\limits_0^{\pi/2} f\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx = \left|\begin{array}{cc} t = \frac{\pi}{2} - x & -dt = dx \\ x = 0 \to t = \frac{\pi}{2} & x = \frac{\pi}{2} \to t \to 0 \end{array}\right| = \int\limits_0^{\pi/2} f(\sin t)\,dt.$$