

PART I: Multiple Choice Questions. No partial marks. Circle the correct answer.

[3] 1. What is the orthogonal trajectories of the family of curves $y = ke^x$?

(a) $y^2 - x = c$ (b) $y^2 + 2x = c$ (c) $y + x^2 = c$ (d) $y + 2x^2 = c$ (e) $y^2 + x^2 = c$

[3] 2. If y is the solution of the initial value problem $y' = 2xy$, $y(0) = 1$, what is $y(1)$?

(a) e (b) e^{-1} (c) $-e$ (d) e^2 (e) $2e$

[3] 3. Let $y'' - 6y' + 8y = e^{4x}$. Which of the following is an appropriate form of a particular solution y_p when using the method of undetermined coefficients?

(a) $y_p = Ae^{4x}$

(b) $y_p = Axe^{4x}$

(c) $y_p = Ax^2e^{4x}$

(d) $y_p = (Ax + B)e^{4x}$

(e) $y_p = Ae^{4x} + Be^{-4x}$

[3] 4. What is the general solution of $x^2y'' + 5xy' + 5y = 0$, $x > 0$?

(a) $y = x^{-1} [c_1 \cos(\ln x) + c_2 \sin(\ln x)]$

(b) $y = x^{-1} [c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)]$

(c) $y = x^{-2} [c_1 \cos(\ln x) + c_2 \sin(\ln x)]$

(d) $y = e^{-2x} [c_1 \cos(x) + c_2 \sin(x)]$

(e) $y = e^{-x} [c_1 \cos(\ln x) + c_2 \sin(\ln x)]$

[3] 5. If y is the solution of the initial-value problem $y'' - 10y' + 25y = 0$, $y(0) = 1$, $y'(0) = 6$, then what is $y(1)$?

- (a) e^5 (b) $2e^5$ (c) $5e^5$ (d) $e^5 + e^{-5}$ (e) e^{-5}

[3] 6. What is the sum of the series $\sum_{n=0}^{\infty} \frac{4 \cdot 3^n}{5^{n+1}}$?

- (a) 1 (b) 2 (c) $\frac{1}{2}$ (d) $\frac{6}{5}$ (e) $\frac{5}{6}$

[3] 7. What is the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n 3^n}$?

- (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 3 (e) ∞

[3] 8. You are given that the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{\sqrt{n}}$ is $R = 1$. What is the interval of convergence?

- (a) $(-1, 1)$ (b) $[-1, 1)$ (c) $(-1, 1]$ (d) $(0, 2)$ (e) $(0, 2]$

[3] 9. Which of the following is a power-series representation of the function $f(x) = \frac{x}{1-x^4}$ about the point $a = 0$?

- (a) $\sum_{n=0}^{\infty} x^{4n+1}$ (b) $\sum_{n=0}^{\infty} (-1)^n x^{4n+1}$ (c) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ (d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ (e) $\sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$

[3] 10. What is the coefficient of x^3 in the Binomial series of the function $f(x) = \frac{1}{(1+x)^4}$ about the point $a = 0$?

- (a) 120 (b) -120 (c) 20 (d) -20 (e) 10

PART II: Long answer questions. Show all your work.

[6] 1. Solve the initial-value problem $y' - 2xy = 2x e^{x^2}$, $y(0) = 3$.

[6] 2. Find the general solution of the equation $y' - \frac{1}{x}y = y^2$.

[6] 3. Find the general solution of the equation $y' = \frac{xy + 3y^2}{x^2 + xy}$.

[10] 4(a) Find the general solution of the equation $3x^2y^2 + 8xy^5 + (2x^3y + 20x^2y^4)y' = 0$.

(b) Show that the differential equation $3xy + 2y^2 + (x^2 + 2xy)y' = 0$ is not exact, and find an integrating factor which makes it exact. Write down the new exact differential equation but do not solve it.

[10] 5(a) Find two linearly independent solutions of the equation $y'' - 6y' + 9y = 0$.

(b) Find a particular solution of the equation $y'' - 6y' + 9y = \frac{e^{3x}}{x^4}$, $x > 0$ by using the method of variation of parameters.

(c) Write down the general solution of $y'' - 6y' + 9y = \frac{e^{3x}}{x^4}$, $x > 0$.

[12] 6. For each one of the following series, determine whether it converges absolutely, converges conditionally, or diverges. Justify your answer.

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^4 + n^3 + n}} \quad (b) \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{4^n} \quad (c) \sum_{n=2}^{\infty} \frac{\ln n}{n}$$

[6] 7. Find the Taylor series of the function $f(x) = e^x$ about the point $a = 3$.

[6] 8. Find the Taylor series of the function $f(x) = \ln(1 + x)$ about the point $a = 0$.

[8] 9. Solve the initial-value problem $y'' + xy' + y = 0$, $y(0) = 0$, $y'(0) = 1$.

Answers to the multiple choice questions

- | | |
|--------|---------|
| 1. (b) | 6. (b) |
| 2. (a) | 7. (d) |
| 3. (b) | 8. (e) |
| 4. (c) | 9. (a) |
| 5. (b) | 10. (d) |

Answers to the long answer questions

- $y = e^{x^2}(x^2 + 3)$.
- $y = \frac{x}{c - x^2/2}$.
- $-\frac{x}{2y} + \frac{1}{2} \ln \left| \frac{y}{x} \right| = \ln |x| + c$.
- (a) $x^3y^2 + 4x^2y^5 = c$.
- (b) $I(x) = x$. New equation is $3x^2y + 2xy^2 + (x^3 + 2x^2y)y' = 0$.
- (a) $y_1 = e^{3x}$, $y_2 = xe^{3x}$.
- (b) $y_p = \frac{e^{3x}}{6x^2}$.
- (c) $y = c_1e^{3x} + c_2xe^{3x} + \frac{e^{3x}}{6x^2}$.
- (a) converges conditionally (b) converges absolutely (c) diverges.
- $\sum_{n=0}^{\infty} \frac{e^3}{n!} (x-3)^n$.
- $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.
- $y = \sum_{k=0}^{\infty} \frac{(-1)^k}{3 \cdot 5 \cdot 7 \cdots (2k+1)} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} x^{2k+1}$.