MATH1005 — Tutorial 6

1. Find the radius and the interval of the convergence of $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n}$.

   Solution: Let $a_n = \frac{n(x-1)^n}{2^n}$.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(n+1)(x-1)^{n+1}}{2^{n+1}} / \frac{n(x-1)^n}{2^n}| = \lim_{n\to\infty} \frac{1}{2}\frac{n+1}{n}|x-1|$$

$$= \frac{1}{2}|x-1| = L.$$

   We use Ratio Test. When $L < 1$, we have $|x-1| < 2$. The series is (absolutely) convergent; When $L > 1$, we have $|x-1| > 2$. The series is divergent. Hence

$$R = 2.$$

   When $x - 1 = -2$, the series is $\sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n n$, which is divergent by Divergence Test.

   When $x - 1 = 2$, the series is $\sum_{n=1}^{\infty} \frac{n(2)^n}{2^n} = \sum_{n=1}^{\infty} n$, which is divergent by Divergence Test.

   Thus the interval of convergence $I = (-1, 3)$.

2. Find the Maclaurin series of e^{5x}.

   Solution: $e^{5x} = 1 + \frac{5x}{1!} + \frac{5^2x^2}{2!} + \frac{5^3x^3}{3!} + ...,\ R = \infty;$

3. Let $\sum_{n=0}^{\infty} c_n(x-1)^n$ be the Taylor series of $f(x) = \sin 6x$, centered at $a = 1$, find c_3.

   Solution: $f'(x) = 6\cos 6x, f''(x) = -6^2 \sin 6x, f'''(x) = -6^3 \cos 6x$. Thus

$$c_3 = f'''(1)/3! = -36\cos 6.$$

4. Let $\sum_{n=0}^{\infty} c_n x^n$ be the binomial series of $f(x) = \frac{1}{\sqrt{4+x^2}}$, find c_4.

Solution:

$$f(x) = \frac{1}{\sqrt{4+x^2}} = \frac{1}{\sqrt{4}}\frac{1}{\sqrt{1+(x/2)^2}} = \frac{1}{2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^{3n+1}n!}x^{2n}\right).$$

c_4 is the coefficient of x^4, where $n=2$.

$$c_4 = \frac{1}{2}\frac{(-1)^2 1\cdot 3}{2^{3(2)+1}2!} = \frac{3}{2^9}.$$

5. Find the Taylor Series of $\frac{\sin(x)}{x}$ near $x=0$;

Solution:

$$\frac{\sin(x)}{x} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}}{x} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} \quad x \neq 0.$$

6. Which of the following is the coefficient of $(x+1)^2$ in the Taylor series of $f(x) = \ln(4+2x)$ about $a=-1$ (center at -1)?

(a) -2 (b) $-1/2$ (c) $-3/4$ (d) 6 (e) -1

Solution: (b).

$$f'(x) = 2(4+2x)^{-1}, f''(x) = -2^2(4+2x)^{-2}, f'''(x) = (-1)(-2)2^3(4+2x)^{-3},$$

$$f^{(4)}(x) = (-1)(-2)(-3)2^4(4+2x)^{-4}, \cdots$$

$$\frac{f''(-1)}{2!} = \frac{-2^2 2^{-2}}{2!} = -\frac{1}{2}.$$

7. Let $f(x)$ be 2π-periodic such that

$$f(x) = \begin{cases} -1, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in [0, \pi). \end{cases}$$

a) Find the Fourier coefficients for this function.

b) List the first two non-zero terms of the Fourier series.

Solution: a) Since $f(x)$ is odd, $f(x)\cos(nx)$ is also odd, thus $a_0 = a_n = 0$.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} 1\sin(nx)dx = -\frac{2}{n\pi}\cos(nx)|_0^{\pi} \\
&= \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases}
\end{aligned}
$$

b) The first two non-zero terms of the Fourier series:

$$b_1 \sin x + b_3 \sin 3x = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x.$$

8. Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1. \\ 0, & 1 \le x < 2; \end{cases}$.

(i) Let $f_{odd}(x)$ be the 4-periodic **odd** extension of $f(x)$. Find the expression of $f_{odd}(x)$ when $-1 < x < 0$.

Solution:

$$f_{odd}(x) = -f(-x) = -(1-(-x)) = -(1+x) = -1-x.$$

(ii) Let $f_{even}(x)$ be the 4-periodic **even** extension of $f(x)$. Find the expression of $f_{even}(x)$ when $-1 < x < 0$.

Solution:

$$f_{even}(x) = f(-x) = (1-(-x)) = 1+x.$$

(iii) Find the Fourier sine series.

Solution: Fourier sine series: for $n = 1, 2, 3, \ldots,$

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\sin(\frac{n\pi x}{2})\,dx \\
&= \int_0^1 (1-x)\sin(\frac{n\pi x}{2})\,dx \\
&= \left[-\frac{2}{n\pi}(1-x)\cos(\frac{n\pi x}{2}) - \frac{4}{n^2\pi^2}\sin(\frac{n\pi x}{2})\right]_0^1 \\
&= \frac{2}{n\pi} - \frac{4}{n^2\pi^2}\sin(\frac{n\pi}{2}).
\end{aligned}
$$

$$f(x) = \sum_{n=1}^{\infty} \left\{ \frac{2}{n\pi} - \frac{4}{n^2\pi^2} \sin(\frac{n\pi}{2}) \right\} \sin(\frac{n\pi x}{2}).$$

(iv) Find a_0 and a_n of the Fourier cosine series.

Solution:

$$\begin{aligned} a_0 &= \frac{2}{L} \int_0^L f(x)\, dx = \frac{2}{2} \int_0^2 f(x)\, dx \\ &= \int_0^1 (1-x)\, dx = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi x}{L})\, dx = \frac{2}{2} \int_0^2 f(x) \cos(\frac{n\pi x}{2})\, dx \\ &= \int_0^1 (1-x) \cos(\frac{n\pi x}{2})\, dx \\ &= \left[\frac{2}{n\pi}(1-x) \sin(\frac{n\pi x}{2}) - \frac{4}{n^2\pi^2} \cos(\frac{n\pi x}{2}) \right]_0^1 \\ &= \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2} \cos(\frac{n\pi}{2}). \end{aligned}$$