MATH1005 — Tutorial 4

1. Show that the sequence $\left\{\sqrt{n^2+6n+1}-\sqrt{n^2-1}\right\}$ converges to 3.

Solution:

$$\sqrt{n^2+6n+1}-\sqrt{n^2-1}=\frac{(\sqrt{n^2+6n+1}-\sqrt{n^2-1})(\sqrt{n^2+6n+1}+\sqrt{n^2-1})}{\sqrt{n^2+6n+1}+\sqrt{n^2-1}}$$

$$=\frac{(n^2+6n+1)-(n^2-1)}{\sqrt{n^2+6n+1}+\sqrt{n^2-1}}=\frac{6n+2}{\sqrt{n^2+6n+1}+\sqrt{n^2-1}}=\frac{6+\frac{2}{n}}{\sqrt{1+\frac{6}{n}+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}}$$

$$\to\frac{6}{2}=3.$$

2. Given the sequence $\{a_n\}$: $a_1=1$, $a_{n+1}=\frac{1}{3}(a_n+4)$.

(a) Show that the sequence is increasing and bounded above.
(b) Find the limit of the sequence.

Solution: (a) (i) We will show that $a_n<2$ by induction: Note that $a_1<2$. Assume that $a_n<2$. Then $a_{n+1}=\frac{1}{3}(a_n+4)<\frac{1}{3}(2+4)=2$.
(ii) $a_{n+1}-a_n=\frac{4}{3}-\frac{2}{3}a_n>\frac{4}{3}-\frac{2}{3}(2)=0$. Thus the sequence is increasing.

(b) By Monotonic Sequence Theorem, the sequence is convergent. Let the limit be L. Then by the recursive relation, $L=\frac{1}{3}(L+4)\Rightarrow L=2$.

3. Calculate the sum of the following series:

$$\sum_{n=1}^{\infty}\frac{2^n+3^n}{5^n}.$$

Solution:

$$\sum_{n=1}^{\infty}\frac{2^n+3^n}{5^n}=\sum_{n=1}^{\infty}\frac{2^n}{5^n}+\sum_{n=1}^{\infty}\frac{3^n}{5^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{\frac{2}{5}}{1-\frac{2}{5}} + \frac{\frac{3}{5}}{1-\frac{3}{5}} = \frac{13}{6}.$$

4. Determine if the series converges or diverges: $\displaystyle\sum_{n=0}^{\infty} \frac{2n+2}{3n+1}$.

   Solution: Since the general term tends to $2/3 \neq 0$, the series is divergent by Divergence Thm.

5. Determine if the series converges or diverges: $\displaystyle\sum_{n=5}^{\infty} \frac{1}{(n+1)(n+3)}$.

   Solution: By Partial Fraction Method,

$$\sum_{n=5}^{\infty} \frac{1}{(n+1)(n+3)} = \sum_{n=5}^{\infty} \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{8} + \frac{1}{7} - \frac{1}{9} + \frac{1}{8} - \frac{1}{10} + \cdots\right) = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{7}\right) = \frac{13}{84}.$$

6. Determine if the following series is convergent or divergent by using Integral Test:

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

   Solution: We use Integral Test. Let $f(x) = xe^{-x^2}, x \geq 1$. Then $f(x)$ is continuous and positive. Note that $f'(x) = e^{-x^2}(1-2x^2) < 0$, hence $f(x)$ is decreasing.

$$\int_1^{\infty} xe^{-x^2} dx = \lim_{t\to\infty} \int_1^t xe^{-x^2} dx$$

$$= \lim_{t\to\infty} \int_1^{t^2} \frac{1}{2} e^{-u} du = \lim_{t\to\infty} \frac{1}{2}(-e^{-t^2} + e^{-1}) = \frac{1}{2e}.$$

   By Integral Test, the given series is convergent.