MATH1005 — Tutorial 3

1. Using variation of parameters find a particular solution to the following equation: $y'' + y = e^x + x^3$.

   Solution: 1) Find two linearly independent solutions to the corresponding homogeneous equation:
$$y'' + y = 0.$$
   Note that the indicial (auxiliary) equation is
$$r^2 + 1 = 0, \Rightarrow r = \pm i, \Rightarrow y_1(x) = \cos x, y_2(x) = \sin x.$$
   2) Let $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. From $y_1(x) = \cos x, y_2(x) = \sin x,\ f(x) = e^x + x^3$ we have
$$u_1'(x) = \frac{-y_2 f}{y_1y_2' - y_1'y_2} = \frac{-\sin x(e^x + x^3)}{\cos^2 x + \sin^2 x} = -\sin x(e^x + x^3), \Rightarrow$$
$$u_1(x) = \frac{1}{2}e^x(\cos x - \sin x) + x^3\cos x - 3x^2\sin x - 6x\cos x + 6\sin x.$$
$$u_2'(x) = \frac{y_1 f}{y_1y_2' - y_1'y_2} = \frac{\cos x(e^x + x^3)}{\cos^2 x + \sin^2 x} = \cos x(e^x + x^3), \Rightarrow$$
$$u_2(x) = \frac{1}{2}e^x(\cos x + \sin x) + x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x.$$
$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = \frac{1}{2}e^x + x^3 - 6x.$$

2. Solve $y'' + 2x(y')^{1/2} = 0$.

   Solution: Since y does not appear explicitly in the equation, let $z = y'$. Then $y'' = z'$, and the equation becomes $z' + 2xz^{1/2} = 0$, which is first-order and separable. Thus,
$$z^{-1/2}dz = -2xdx, \Rightarrow 2z^{1/2} = -x^2 + c, \Rightarrow 4z = x^4 - 2cx^2 + c^2, \Rightarrow$$
$$4y' = x^4 - 2cx^2 + c^2, \Rightarrow 2y^2 = \frac{1}{5}x^5 - \frac{2}{3}cx^3 + c^2x + d.$$

3. Solve $y'' - 2yy' = 0$, $y(0) = 1$, $y'(0) = 2$.

Solution: Since x does not appear explicitly in the equation, put $y'(x) = u(y)$. Then $y'' = \frac{du}{dy}\frac{dy}{dx} = uu'$, and the equation becomes $u(u' - 2y) = 0$. Thus, either

$$u = 0, \Rightarrow y' = 0, \Rightarrow y = const;$$

or

$$u' - 2y = 0 \Rightarrow du = 2ydy \Rightarrow u = y^2 + c \Rightarrow$$

$$u(y(0)) = y(0)^2 + c, \Rightarrow y'(0) = y(0)^2 + c, \Rightarrow c = 1, \Rightarrow u = y^2 + 1 \Rightarrow$$

$$\frac{dy}{1+y^2} = dx, \Rightarrow \arctan y = x + d, \Rightarrow \arctan y(0) = d, \Rightarrow d = \arctan 1 = \frac{\pi}{4}, \Rightarrow$$

$$\arctan y = x + \frac{\pi}{4}.$$

4. Solve the DE: $y''' - 3y'' - y' + 3y = 0$.

Solution: The indicial equation is $r^3 - 3r^2 - r + 3 = 0 \Rightarrow r = 1, -1, 3$. Thus

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{3x}.$$

5. Solve the DE: $y''' - 5y'' + 8y' - 4y = 0$.

Solution: The indicial equation is $r^3 - 5r^2 + 8r - 4 = 0 \Rightarrow r = 1, 2, 2$. Thus

$$y = c_1 e^x + (c_2 + c_3 x)e^{2x}.$$

6. Solve the DE: $y''' + 4y'' + 6y' + 4y = 0$.

Solution: The indicial equation is $r^3 + 4r^2 + 6r + 4 = 0 \Rightarrow r = -2, -1 \pm i$. Thus

$$y = c_1 e^{-2x} + (c_2 \cos x + c_3 \sin x)e^{-x}.$$

7. Find the general solution: $\begin{cases} x' &= 2x + y \\ y' &= 5x - 2y \end{cases}.$

Solution: In matrix form, the system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$.

$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 5 & -2-\lambda \end{vmatrix} = \lambda^2 - 9 = (\lambda-3)(\lambda+3) = 0 \Rightarrow \lambda_1 = 3,\ \lambda_2 = -3.$

For $\lambda_1 = 3$, $\begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0} \Rightarrow -a + b = 0,\ a = 1 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -3$, $\begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0} \Rightarrow 5a + b = 0,\ a = 1 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

Thus, $\mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2(t) = e^{-t} \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ are independent solutions, and the general solution is $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, i.e.,

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-t} \\ y(t) &= c_1 e^t - 5c_2 e^{-t} \end{aligned}.$$

8. Find the general solution: $\begin{cases} x' = 6x - 2y \\ y' = 8x - 2y \end{cases}$.

Solution: In matrix form, the system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 6 & -2 \\ 8 & -2 \end{bmatrix}$.

$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -2 \\ 8 & -2-\lambda \end{vmatrix} = (\lambda - 2)^2 \Rightarrow \lambda_1 = \lambda_2 = 2.$

$(A - 2I) \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0} \Rightarrow 2a - b = 0,\ a = 1 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solving $(A - 2I) \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\Rightarrow 4c - 2d = 1, \Rightarrow c = 1, d = 1.5 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$.

Thus, $\mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2(t) = te^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$ are independent solutions, and the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = e^{2t}\begin{bmatrix} c_1 + c_2 t + c_2 \\ 2c_1 + 2c_2 t + 1.5c_2 \end{bmatrix}.$$