MATH1005 — Tutorial 2

1. Given that $y_1(x) = e^x$ is a solution of the DE $y'' - 3y' + 2y = 0$. Find a second linearly independent solution.

   Solution:
$$u(x) = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\,dx = \int \frac{1}{(e^x)^2} e^{-\int -3\,dx}\,dx = \int e^x\,dx = e^x.$$
$$y_2(x) = y_1(x) \int u(x)\,dx = e^{2x}.$$

2. Given that $y_1 = x^2$ is a solution of
$$y'' - \frac{7}{x}y' + \frac{12}{x^2}y = 0,\ x > 0.$$
   Which of the following is a second linearly independent solution?

   (a) $y = x^2 \ln|x|$ (b) $y = x^3$ (c) x^4 (d) x^5 (e) x^6

   Solution: (e).
$$u(x) = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\,dx = \int \frac{1}{x^4} e^{-\int -7/x\,dx}\,dx = \int \frac{1}{x^4} e^{7\ln x} dx = \int \frac{x^7}{x^4} dx$$
$$= \int x^3 dx = \frac{1}{4}x^4.$$
   Thus $y_2(x) = y_1(x)u(x) = \frac{1}{4}x^6$.

3. Solve the boundary-value problem, if possible :
$$y'' - 3y' + 2y = 0, \quad y(0) = 1, y(3) = 0.$$
   Solution: The indicial equation (or auxiliary equation) is
$$r^2 - 3r + 2 = 0, \Rightarrow r = 1, 2.$$

So the general solution is

$$y(x) = C_1 e^x + C_2 x e^{2x}.$$

From $y(0) = 1, y(3) = 0$ we have

$$C_1 e^0 + C_2 e^0 = 1, C_1 e^3 + C_2 e^{2(3)} = 0, \Rightarrow C_1 = \frac{e^3}{e^3 - 1}, C_2 = \frac{1}{1 - e^3}.$$

The solution is

$$y(x) = \frac{e^3}{e^3 - 1} e^x + \frac{1}{1 - e^3} e^{2x}.$$

4. Solve the boundary-value problem, if possible :

$$y'' + 4y' + 13y = 0, \quad y(0) = 2, y(\pi/2) = 1.$$

Solution: The indicial equation (or auxiliary equation) is

$$r^2 + 4r + 13 = 0, \Rightarrow r = -2 \pm 3i.$$

So the general solution is

$$y(x) = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x).$$

From $y(0) = 2, y(\pi/2) = 1$ we have

$$C_1 = 2, e^{-\pi} C_2(-1) = 1, \Rightarrow C_1 = 2, C_2 = -e^{\pi}.$$

The solution is

$$y(x) = e^{-2x}(2 \cos 3x - e^{\pi} \sin 3x).$$

5. Solve $x^2 y'' + 2xy' - 6y = 0,\ x > 0.$

Solution: The indicial equation is

$$r^2 + r - 6 = 0 \ \Rightarrow\ (r + 3)(r - 2) = 0 \ \Rightarrow y = c_1 x^{-3} + c_2 x^2 = \frac{c_1}{x^3} + c_2 x^2.$$

6. Solve $x^2y'' + 9xy' + 16y = 0,\ x > 0.$

Solution: The indicial equation is

$$r^2 + 8r + 16 = 0 \ \Rightarrow\ (r+4)^2 = 0 \ \Rightarrow y = \frac{c_1}{x^4} + c_2\frac{\ln(x)}{x^4}.$$

7. Solve $x^2y'' - 3xy' + 8y = 0,\ x > 0.$

Solution: The indicial equation is

$$r^2 - 4r + 8 = 0 \ \Rightarrow\ r = 2 \pm 2i \ \Rightarrow$$

$$y = x^2\left[c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))\right].$$

8. Solve the initial-value problem using the method of undetermined coefficients:

$$y'' + y = e^x + x^3, \quad y(0) = 2, y'(0) = 0.$$

Solution: 1) Find the general solution of

$$y'' + y = 0.$$

Note that the auxiliary equation is

$$r^2 + 1 = 0, \Rightarrow r = \pm i, \Rightarrow y_c(x) = C_1\cos x + C_2\sin x.$$

2) For the equation $y'' + y = e^x$, we try $y_{p_1}(x) = Ae^x, \Rightarrow y'_{p_1}(x) = Ae^x$, $y''_{p_1}(x) = Ae^x$. We have

$$2A = 1, \Rightarrow A = \frac{1}{2}, \Rightarrow y_{p_1}(x) = \frac{1}{2}e^x.$$

3) For the equation $y'' + y = x^3$, we try $y_{p_2}(x) = Ax^3 + Bx^2 + Cx + D$. Then

$$y'_{p_2}(x) = 3Ax^2 + 2Bx + C, \quad y''_{p_2}(x) = 6Ax + 2B. \Rightarrow$$

$$y''_{p_2}(x) + y_{p_2}(x) = 6Ax + 2B + Ax^3 + Bx^2 + Cx + D = Ax^3 + Bx^2 + (6A + C)x + 2B + D \Rightarrow$$

$$Ax^3 + Bx^2 + (6A + C)x + 2B + D = x^3, \Rightarrow$$

$$A = 1, B = 0, 6A + C = 0, 2B + D = 0 \Rightarrow A = 1, B = 0, C = -6, D = 0 \Rightarrow$$

$$y_{p_2}(x) = x^3 - 6x.$$

4) By the superposition principle, the general solution is

$$y(x) = y_c(x) + y_{p_1}(x) + y_{p_2}(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2}e^x + x^3 - 6x.$$

5) Solve C_1 and C_2: From $y(0) = 2$ we have

$$C_1 + \frac{1}{2} = 2, \Rightarrow C_1 = \frac{3}{2}.$$

Since

$$y'(x) = -C_1 \sin x + C_2 \cos x + \frac{1}{2}e^x + 3x^2 - 6.$$

From $y'(0) = 0$ we have

$$C_2 + \frac{1}{2} - 6 = 0, \Rightarrow C_2 = \frac{11}{2}, \Rightarrow$$

$$y(x) = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x.$$

9. Write a trial solution for the method of undetermined coefficients. Do not solve the coefficients:

$$y'' + 2y' + 10y = x^2 e^{-x} \cos 3x.$$

Solution: 1) Find the general solution of

$$y'' + 2y' + 10y = 0.$$

Note that the auxiliary equation is

$$r^2 + 2r + 10 = 0, \Rightarrow r = -1 \pm 3i, \Rightarrow y_c(x) = e^{-x}(C_1 \cos 3x + C_2 \sin 3x).$$

2) To find $y_p(x)$, we try

$$y_p(x) = xe^{-x}[(Ax^2 + Bx + C)\cos 3x + (Dx^2 + Ex + F)\sin 3x]$$