MATH1005 — Tutorial 1

1. Solve
$$\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{ye^y}.$$
Solution: By separating x and y we have
$$ye^y dy = x\sqrt{x^2+1}dx \Rightarrow \int ye^y dy = \int x\sqrt{x^2+1}dx.$$
By Integration-by-pats,
$$\int ye^y dy = ye^y - e^y + C.$$
Let $u = x^2+1$, then $du = 2xdx$. Thus
$$\int x\sqrt{x^2+1}dx = \int \sqrt{u}\frac{du}{2} = \frac{4}{3}u^{3/2} + C = \frac{4}{3}(x^2+1)^{3/2} + C.$$
The solution is
$$ye^y - e^y = \frac{4}{3}(x^2+1)^{3/2} + C.$$

2. Solve IVP:
$$\frac{dy}{dx} = y^2 + 1 \quad y(0) = 1.$$
Solution: By separating x and y we have
$$\frac{dy}{y^2+1} = dx, \Rightarrow \arctan y = x + C.$$
Note that $y(0) = 1$, $\arctan 1 = 0 + C, \Rightarrow C = \frac{\pi}{4}$. Thus $y = \tan(x + \frac{\pi}{4})$.

3. Solve the DE: $y' - 3y = e^x$.

Solution: 1) Find the integrating factor I(x)
$$I(x) = e^{\int -3dx} = e^{-3x}.$$
2) Multiply the original equation by $I(x)$ we have
$$e^{-3x}(y' - 3y) = e^{-3x}e^x, \Rightarrow (e^{-3x}y)' = e^{-2x}.$$
3) Integrate the two sides:
$$e^{-3x}y = \int e^{-2x}dx = -\frac{1}{2}e^{-2x} + C, \Rightarrow y = -\frac{1}{2}e^x + Ce^{3x}.$$

4. Solve the IVP: $y' + y = x + e^x, y(0) = 0$.

Solution: 1) Find the integrating factor I(x)

$$I(x) = e^{\int 1dx} = e^x.$$

2) Multiply the original equation by $I(x)$ we have

$$e^x(y' + y) = e^x(x + e^x), \Rightarrow (e^x y)' = xe^x + e^{2x}.$$

3) Integrate the two sides:

$$e^x y = \int [xe^x + e^{2x}]dx = xe^x - e^x + \frac{1}{2}e^{2x} + C, \Rightarrow y = x - 1 + \frac{1}{2}e^x + Ce^{-x}.$$

5. Let $y(x)$ be the solution of the IVP:

$$\frac{dy}{dx} = 2xy^2, \quad y(0) = 1.$$

Find $y(2)$.

(a) 1 (b) $-1/3$ (c) $1/2$ (d) -1 (e) -2

Solution: (b). By separating x and y we have

$$\frac{dy}{y^2} = 2xdx, \Rightarrow -\frac{1}{y} = x^2 + C.$$

Note that $y(0) = 1$, $-\frac{1}{1} = 0 + C, \Rightarrow C = -1$. Thus $-\frac{1}{y} = x^2 - 1$, $-\frac{1}{y(2)} = 3, \Rightarrow$ $y(2) = -1/3$.

6. Solve the homogeneous DE:

$$xy' = y + xe^{y/x}.$$

(a) $y = -x \ln|x| + C$ (b) $y = x \ln|x| + C$ (c) $-y/x = \ln|x| + C$

(d) $e^{-y/x} = \ln|x| + C$ (e) $-e^{-y/x} = \ln|x| + C$

Solution: (e). Divide the two sides by x we get $y' = y/x + e^{y/x}$. Let $u = \frac{y}{x}$. Then $y' = (xu)' = u + xu'$. We have

$$u + xu' = u + e^u, \Rightarrow xu' = e^u, \Rightarrow e^{-u}du = \frac{1}{x}dx, \Rightarrow -e^{-u} = \ln|x| + C, \Rightarrow -e^{-y/x} = \ln|x| + C.$$

7. Solve the following DE:

$$y' + \frac{2}{x}y = \frac{y^3}{x^2}.$$

Solution: This is a Bernoulli DE with $n = 3$.

1) Let $u = y^{1-n} = y^{-2}$. Then $u' = (1-n)y^{-n}y' = -2y^{-3}y', \Rightarrow y' = -\frac{1}{2}y^3u'$.

2) Substitute this into the DE we have

$$-\frac{1}{2}y^3u' + \frac{2}{x}y = \frac{y^3}{x^2}, \Rightarrow -\frac{1}{2}u' + \frac{2}{x}y^{-2} = \frac{1}{x^2},$$

$$\Rightarrow -\frac{1}{2}u' + \frac{2}{x}u = \frac{1}{x^2}, \Rightarrow$$

$$u' - \frac{4}{x}u = -\frac{2}{x^2}. \tag{1}$$

3) Now we have a linear DE.

$$I(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = \frac{1}{x^4}.$$

4) Multiply (1) by $I(x)$:

$$\frac{1}{x^4}(u' - \frac{4}{x}u) = -\frac{2}{x^2}\frac{1}{x^4} \Rightarrow (\frac{1}{x^4}u)' = -\frac{2}{x^6}.$$

5) Integrating this we have

$$\frac{u}{x^4} = \frac{2}{5}x^5 + C, \Rightarrow u = \frac{2x}{5} + C, \Rightarrow y^{-2} = \frac{2x}{5} + C.$$

8. Find an integrating factor and thus solve the equation:

$$1 - xy + x(y - x)y' = 0, \quad x > 0.$$

Solution: Let $P(x,y) = 1 - xy$, $Q(x,y) = x(y-x)$. Then $P_y = -x$, $Q_x = y - 2x$. $P_y \neq Q_x$. The equation is not exact.

$$\frac{dI}{dx} = \frac{P_y - Q_x}{Q}I = \frac{x - y}{x(y - x)}I = -\frac{1}{x}I. \Rightarrow$$

$$\frac{dI}{I} = -\frac{1}{x}dx, \Rightarrow I(x) = \frac{1}{x}.$$

There exists $f(x,y)$ such that

$$f_x(x,y) = I(x)P(x,y) = \frac{1}{x} - y, \quad (1)$$

and

$$f_y(x,y) = I(x)Q(x,y) = y - x. \quad (2)$$

Integrating (1) to x we have

$$f(x,y) = \int (\frac{1}{x} - y)dx + g(y) = \ln x - xy + g(y). \Rightarrow$$

$$f_y(x,y) = -x + g'(y).$$

Combining this with (2) we have $g'(y) = y. \Rightarrow g(y) = \frac{1}{2}y^2. \Rightarrow$

$$f(x,y) = \ln x - xy + \frac{1}{2}y^2 + C.$$

The solution is

$$\ln x - xy + \frac{1}{2}y^2 = C.$$