MATH1005 — Review

First-order Differential Equations

1. Separable DE: $\frac{dy}{dx} = f(x)g(y), \Rightarrow dy/g(y) = f(x)dx$.

2. Homogeneous eqn: $\frac{dy}{dx} = g(\frac{y}{x})$. Let $v = \frac{y}{x}$. Then it is separable.

3. 1st order linear eqn: $\frac{dy}{dx} + P(x)y = Q(x)$. The solution is $y(x)I(x) = \int I(x)Q(x)dx + C$, where the integrating factor $I(x) = e^{\int P(x)dx}$.

4. Bernoulli eqn: $\frac{dy}{dx} + P(x)y = Q(x)y^n$. Let $u = y^{1-n}$, then $\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$, which is a 1st order linear eqn.

5. Exact eqn: $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$ or $P(x,y)dx + Q(x,y)dy = 0$ with $P_y = Q_x$. The solution is $f(x,y) = C$, where $f_x = P$, $f_y = Q$.

6. Non-exact eqn: $P(x,y)dx + Q(x,y)dy = 0$ with $P_y \neq Q_x$. Change the eqn to exact by multiplying integrating factor I:

   - $I = \exp\left\{\int \frac{P_y - Q_x}{Q}\, dx\right\}$, if $\frac{P_y - Q_x}{Q}$ is a function of x only;
   - $I = \exp\left\{-\int \frac{P_y - Q_x}{P}\, dy\right\}$, if $\frac{P_y - Q_x}{P}$ is a function of y only.

Second-order Differential Equations

1. Homogeneous linear DE with constant coefficients: $ay'' + by' + cy = 0$. Let r_1 and r_2 be two solutions of the indicial eqn $ar^2 + br + c = 0$.

   - $r_1 \neq r_2$: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
   - $r_1 = r_2$: $y = (C_1 + C_2 x)e^{rx}$.
   - $r = \alpha + i\beta$: $y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

2. Reduction of order: If y_1 is a solution of $y'' + p(x)y' + q(x)y = 0$. Then $y_2 = u(x)y_1$, where $u' = \frac{1}{y_1^2} e^{-\int p(x)dx}$.

3. Non-homogeneous DE:

$$y'' + p(x)y' + q(x)y = G(x).$$

Then $y(x) = y_p(x) + y_c(x)$, where $y_p(x)$ is a particular solution of the equation, $y_c(x)$ is the general solution of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \qquad (1)$$

To find $y_p(x)$,

- Variation of parameters: If $y_c = C_1y_1 + C_2y_2$, then $y_p = u_1y_1 + u_2y_2$, where

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, \\ u_1'y_1' + u_2'y_2' = \frac{G(x)}{a}. \end{cases}$$

- Undetermined coefficients:
  - $G(x) = P(x)$ a polynomial with degree n.
    (a) If x^k is not a solution of (1) for any $k \leq n$, then let $y_p(x)$ be a polynomial with degree n.
    (b) If x^k is a solution of (1) for $k \leq n$, then let $y_p(x) = xQ(x)$, where $Q(x)$ is a polynomial with degree n.
    (c) If x^k and x^s are solutions of (1) for $k \leq n$ and $s \leq n + 1$, then let $y_p(x) = x^2Q(x)$, where $Q(x)$ is a polynomial with degree n.
  - $G(x) = e^{kx}$.
    (a) If e^{kx} is not a solution of (1), then let $y_p(x) = Ae^{kx}$.
    (b) If e^{kx} is a solution of (1), but xe^{kx} not, then let $y_p(x) = Axe^{kx}$.
    (c) If both e^{kx} and xe^{kx} are solutions of (1), then let $y_p(x) = Ax^2e^{kx}$.
  - $G(x) = a\cos kx + b\sin kx$.
    (a) If $\cos kx$ and $\sin kx$ are not solutions of (1), then let $y_p(x) = A\cos kx + B\sin kx$.
    (b) If $\cos kx$ or $\sin kx$ are solutions of (1), then let $y_p(x) = x[A\cos kx + B\sin kx]$.
  - $G(x) = e^{kx}P(x)(s\cos mx + t\sin mx)$.
    Let $y_p(x) = e^{kx}[Q(x)\cos mx + R(x)\sin mx]$, where $\deg Q = \deg R = \deg P$.

- $G(x)$ is a combination of $P(x), e^{kx}, \cos kx, \sin kx$. Then use the Principle of Superposition.

4. Cauchy-Euler eqn: $x^2y''+Axy'+By=0$. The indicial equation is $r^2+(A-1)r+B=0$.
   - If $r_1 \neq r_2$ are real, then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$.
   - If $r_1 = r_2$ (real), then $y_1 = x^{r_1}$ and $y_2 = x^{r_1}\ln(x)$.
   - If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $y_1 = x^\alpha \cos[\beta \ln(x)]$ and $y_2 = x^\alpha \sin[\beta \ln(x)]$.
5. Second-Order Equations Reducible to the First Order
   - $f(x, y', y'') = 0$. The dependent variable (e.g., y) does not appear, let $u(x) = y'(x)$, then $y''(x) = u'(x)$.
   - $f(y, y', y'') = 0$. The independent variable (e.g., x) does not appear, let $y'(x) = u(y)$, then, by the chain rule, $y'' = u\frac{du}{dy}$.

Higher-order homogeneous linear equation with constant coefficients

$$a_k y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1 y' + a_0 y = 0.$$

The indicial equation is

$$a_k r^k + a_{k-1}r^{k-1} + \cdots + a_1 r + a_0 = 0.$$

- If the indicial equation has k different roots r_i, then the solution is

$$y = c_1 e^{r_1 x} + \cdots + c_k e^{r_k x}.$$

- If the indicial equation has a root a with multiplicity m, then one solution is

$$y = e^{ax}(c_1 + c_2 x + \cdots + c_m x^{m-1}).$$

- If the indicial equation has a root $a + bi$ with multiplicity m, then one solution is

$$y = e^{ax}\cos(bx)[c_1 + c_2 x + \cdots + c_m x^{m-1}] + e^{ax}\sin(bx)[d_1 + d_2 x + \cdots + d_m x^{m-1}].$$

2×2 Homogeneous Systems of Equations

Let x and y be functions of t. A system of two equations with constant coefficients has the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy. \end{aligned}$$

Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Set $\det(\lambda I - A) = 0$, and solve for λ. For each eigenvalue λ, the corresponding eigenvectors are the nonzero solutions $\mathbf{v}$ of the matrix equation $(\lambda I - A)\mathbf{v} = 0$.

- If the 2×2 matrix A has two distinct real eigenvalues λ_1, λ_2. Then the matrix A has two independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, corresponding to the eigenvalues λ_1 and λ_2 respectively (distinct or not), then $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2$ are two independent solutions of the matrix equation, with the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, where c_1 and c_2 are arbitrary constants.

- If the 2×2 matrix A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$, and the matrix A has two independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then it is same as Case 1:

$$\vec{x}_1 = e^{\lambda t}\vec{v}_1, \quad \vec{x}_2 = e^{\lambda t}\vec{v}_2.$$

- If the 2×2 matrix A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ with only one independent eigenvector $\mathbf{K}$, then one solution is given by $\mathbf{x}_1 = e^{\lambda t}\mathbf{K}$. A second linearly independent solution is given by

$$\mathbf{x}_2 = te^{\lambda t}K + e^{\lambda t}P,$$

where P satisfies $(A - \lambda I)P = K$.

- If the 2×2 matrix A has complex eigenvalues, $\lambda = a + bi$. Let $\mathbf{K}$ be the corresponding eigenvector. Then

$$X_1 = (B_1 \cos bt - B_2 \sin bt)e^{at}, X_2 = (B_2 \cos bt + B_1 \sin bt)e^{at}, \quad B_1 = Re\mathbf{K}, B_2 = Im\mathbf{K}.$$

Sequences

1. Sequence $a_1, a_2, ..., a_n, ...$, or $\{a_n\}_{n=1}^{\infty}$. If $\lim_{n\to\infty} a_n$ exists, then we say the sequence converges. Otherwise, we say the sequence diverges.
2. Squeeze Theorem: If $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.
3. Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.

Series

1. Partial sum: $S_n = \sum_{j=1}^{n} a_j$. Then $\sum_{j=1}^{\infty} a_j = S \Leftrightarrow \lim_{n\to\infty} S_n = S$.
2. Geometric series: if $|r| < 1$ then $\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$.
3. Partial fraction: $\frac{k}{n(n+k)} = \frac{1}{n} - \frac{1}{n+k}$.
4. Harmonic series $\sum_{j=1}^{\infty} \frac{1}{n}$ is divergent.
5. Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{j=1}^{\infty} a_j$ is divergent.
6. Integral test: If $f(x)$ is continuous, positive, decreasing, $f(j) = a_j$, then

$$\sum_{j=1}^{\infty} a_j \text{ is convergent} \Leftrightarrow \int_1^{\infty} f(x)dx \text{ is convergent.}$$

7. The $p-$series: $\sum_{j=1}^{\infty} \frac{1}{j^p}$ is: convergent if $p > 1$ and divergent if $p \le 1$.
8. Direct comparison test: If $0 \le a_j \le b_j$, then

$$\sum_{j=1}^{\infty} b_j \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

$$\sum_{j=1}^{\infty} a_j \text{ is divergent} \Rightarrow \sum_{j=1}^{\infty} b_j \text{ is divergent.}$$

9. Limit Comparison test: If $a_n > 0$, $b_n > 0$ and

$$\lim \frac{a_n}{b_n} = c > 0,$$

then

$$\sum a_j \text{ is divergent} \Leftrightarrow \sum b_j \text{ is divergent.}$$

10. Alternating series Test: The alternating series $\sum_{j=1}^{\infty}(-1)^{j-1}b_j$ is convergent if

    (a) $b_j > 0$

    (b) b_j decreasing ($b_1 \geq b_2 \geq b_3 \geq \cdots$)

    (c) $\lim_{j\to\infty} b_j = 0$.

11. Absolute and conditional convergence:

$$\sum_{j=1}^{\infty} |a_j| \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

If $\sum_{j=1}^{\infty} a_j$ is convergent, but $\sum_{j=1}^{\infty} |a_j|$ is divergent, then $\sum_{j=1}^{\infty} a_j$ is conditionally convergent.

12. Ratio test: Consider the series $\sum_{j=1}^{\infty} a_j$, with $\lim_{j\to\infty} |\frac{a_{j+1}}{a_j}| = L$.

    - If $L < 1$, then the series is absolutely convergent;
    - If $L > 1$, then the series is divergent;
    - If $L = 1$, no conclusion.

13. Root test: Consider the series $\sum_{j=1}^{\infty} a_j$, with $\lim_{j\to\infty} \sqrt[n]{|a_j|} = L$.

    - If $L < 1$, then the series is absolutely convergent;
    - If $L > 1$, then the series is divergent;
    - If $L = 1$, no conclusion.

Power Series

Consider the series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

- Radius of convergence: Using Ratio Test to find R.
- Interval of convergence: $I =< a-R, a+R >$, symmetric to the center a, with two end points $a-R$ and $a+R$. The convergence or divergence at the two end points $x=a-R$ and $x=a+R$ should be checked.

Taylor Series

1. Taylor polynomial: Taylor polynomial of degree n approximating $f(x)$ for x at a:

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $n = 1$, we have the linear approximation.

2. Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R.$$

3. Mactaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n, \quad |x| < R.$$

4. Series for some special functions such as: e^x, $\sin x$, $\cos x$, $\ln(1+x)$.

5. Binomial series: If p is a real number and $|x| < 1$, then

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \begin{pmatrix} p \\ n \end{pmatrix} x^n,$$

here

$$\begin{pmatrix} p \\ n \end{pmatrix} = \frac{p(p-1)\cdots(p-n+1)}{n!}, \quad \begin{pmatrix} p \\ 0 \end{pmatrix} = 1.$$

Application: Let $f(x) = (1+x)^p$, then

$$f^{(n)}(0) = \begin{pmatrix} p \\ n \end{pmatrix} n! = p(p-1)\cdots(p-n+1).$$

6. Represent functions as Taylor series:

(a) By substitution by using known Taylor series. Basic result:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

(b) Term-by-term differentiation:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \Rightarrow f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

and f and f' have the same radius of convergence.

(c) Term-by-term integration:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \Rightarrow \int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1},$$

and f and $\int f dx$ have the same radius of convergence.

Fourier Series

- The (full) Fourier series of $2L$-periodic function $f(x)$ can be written as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right\},$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx.$$

- Given a function $f(x)$ defined on $(0, L)$. The (half-range) Fourier sine series is

$$\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L})\, dx.$$

- Given a function $f(x)$ defined on $(0, L)$. The (half-range) Fourier cosine series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi x}{L})\, dx.$$

- The Fourier Series of $f(x)$ on (a,b) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})),$$

at every $x \in [a,b]$ where f is continuous. The coefficients a_n $(n \geq 0)$ and the coefficients b_n $(n \geq 1)$ are calculated as follows:

$$a_n = \frac{1}{L}\int_a^b f(x)\cos(\frac{n\pi x}{L})\,dx,\ n = 0,1,2,\ldots; \quad b_n = \frac{1}{L}\int_a^b f(x)\sin(\frac{n\pi x}{L})\,dx,\ n = 1,2,3,\ldots.$$