MATH1005 – Notes — By Eric Hua

Contents

Chapter 1: Introduction

1.1 Basic concepts

What is a differential equation?

First-order DE: $f(y', y, x) = 0$.

Second-order DE: $f(y'', y', y, x) = 0$.

n-th order DE: $f(y^{(n)}, ..., y', y, x) = 0$.

IVP: Initial value problem = DE with initial conditions.
Family of solutions: if a DE without initial condition, then it may have infinite solutions.

Example. Match the following DE and possible solutions.

DE: (a) $y'' = y$ (b) $y' = -y$ (c) $y' = 1/y$ (d) $y'' = -y$ (e) $x^2y'' - 2y = 0$.

SOL: (1) $y = \cos x$ (2) $y = \cos(-x)$ (3) $y = x^2$ (4) $y = e^x + e^{-x}$ (5) $y = \sqrt{2x}$.

Chapter 2: First-order Equations

2.1 Separable equations

An equation is **separable** if it can be written as

$$\frac{dy}{dx} = g(x)f(y),$$

its solution is

$$\int \frac{1}{f(y)} dy = \int g(x)dx.$$

Example 1 *Suppose we have the equation* $\frac{dy}{dt} = \frac{4t}{3y^2+\cos y}$.

(a) Solve the equation.

Solution: This is a separable equation. We rewrite the equation as $(3y^2 + \cos y)dy = 4tdt$. Integrating it gives

$$y^3 + \sin y = 2t^2 + C,$$

where C is a constant.

(b) Find the solution satisfying the initial condition $y(\pi) = 0$.

Solution: From part a) we have

$$0^3 + \sin 0 = 2\pi^2 + C,$$

which implies that $C = -2\pi^2$, and so the special solution is

$$y^3 + \sin y = 2t^2 - 2\pi^2,$$

Example 2 *Solve the following IVP:*

$$\frac{dp}{dt} = (1 + p^2)te^t, \quad p(0) = 1.$$

Solution: We write the equation as

$$\frac{dp}{1+p^2} = te^t dt,$$

$$\int \frac{dp}{1+p^2} = \int te^t dt.$$

Now we apply Integration-By-Parts to the right hand side. we obtain

$$\arctan p = te^t - e^t + c.$$

Since $p(0) = 1$, we have

$$\arctan 1 = 0 - 1 + C, \quad C = \pi/4 + 1.$$

Hence

$$\arctan p = te^t - e^t + \pi/4 + 1, \quad or \quad p = \tan(te^t - e^t + \pi/4 + 1).$$

Example 3 *Solve the following IVP:*

$$\frac{dy}{dt} - y^2 - y^2 t = 0, \quad y(1) = 2.$$

Example 4 *Solve the following differential equation*

$$(p^2 - p - 2)\frac{dp}{dt} = (p^2 - 3p + 2)te^t.$$

Remark. One special solution is $p(t) = 2$.

Example 5 *Solve the Logistic Equation:*

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where k is the relative growth rate, M is the carrying capacity.

Solution: By Partial Fraction,

$$\int \left(\frac{1}{P} - \frac{\frac{1}{M}}{1 - \frac{P}{M}}\right) dP = \int k dt.$$

Its solution is:

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \quad A = \frac{M - P_0}{P_0}.$$

Orthogonal trajectory: Orthogonal trajectories are a family of curves in the plane that intersect a given family of curves at right angles.
To find the Orthogonal trajectories,

1) Calculate y' from $f(x, y) = 0$;
2) Solve the equation $\frac{dy}{dx} = -\frac{1}{y'}$.

Example 6 *Find the Orthogonal trajectories of a family of curves* $y = c/x$.

Example 7 *Find the Orthogonal trajectories of a family of curves* $y = kx^2$.

Example 8 *Find the Orthogonal trajectories of a family of curves* $x^2 + y^2 = r^2$.

2.2 Homogeneous DE

Definition 1 *A function* $z = f(x, y)$ *is said to be a homogeneous function of degree* n, *if* $f(tx, ty) = t^n f(x, y)$.

For example, $f(x, y) = xy^2 + x^3$ is homogeneous function of degree 3.

Definition 2 *A first-order DE* $M(x, y)dx + N(x, y)dy = 0$ *is said to be homogeneous (in* (x, y)*) if both* $M(x, y)$ *and* $N(x, y)$ *are homogeneous functions of the same degree. It is equivalent to*

$$\frac{dy}{dx} = f(\frac{y}{x}).$$

Method to solve homogeneous DE: by substitution $y = ux$**.**

Example 9 *Solve*

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0.$$

Solution: 1) The coefficients are homogeneous functions of degree 2.

2) Let $y = ux$. Then we get a separable DE

$$\frac{1-u}{1+u}du + \frac{1}{x}dx = 0.$$

The solution is

$$-u + 2\ln|1+u| + \ln|x| = \ln|c|.$$

3) Final solution is

$$(x+y)^2 = cxe^{y/x}.$$

2.3 Linear equations

Definition 3 *A first-order differential equation is called* ***linear*** *if it can be written in the form*

$$a(x)\frac{dy}{dx} + b(x)y = c(x).$$

Zeros of $a(x)$ are called singular points of the equation. The **standard form of the first-order linear DE** *is:*

$$\frac{dy}{dx} + P(x)y = Q(x).$$

If $Q(x) = 0$, then it is called homogeneous; otherwise it is non-homogeneous.

Theorem 1 *If y_p is a particular solution of the LDE*

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and y_c is the general solution of the associated homogeneous DE

$$\frac{dy}{dx} + P(x)y = 0,$$

then $y_c + y_p$ is the general solution of $\frac{dy}{dx} + P(x)y = Q(x)$.

Integrating factors

Consider a first-order linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

This equation may be solved by multiplying both sides by the function

$$I(x) = e^{\int P dx},$$

which is called an **integrating factor.** The left side becomes the derivative of $ye^{\int P dx}$. Integration of both sides will yield the solution.

Theorem 2 *The general solution of a first order linear equation is given by*

$$y = e^{-\int P dx}\left[\int e^{\int P dx} Q dx + C\right]$$

where C is an arbitrary constant.

Example 10 *Solve $y' + xy = x$.*

This is a linear first order ODE in standard form with $p(x) = q(x) = x$. The integrating factor is

$$I(x) = e^{\int x dx} = e^{x^2/2}.$$

Hence, after multiplying both sides of our differential equation, we get

$$\frac{d}{dx}(e^{x^2/2}y) = xe^{x^2/2}$$

which, after integrating both sides, yields

$$e^{x^2/2}y = \int xe^{x^2/2}dx = e^{x^2/2} + C.$$

Hence the general solution is $y = 1 + Ce^{-x^2/2}$. The solution satisfying the initial condition $y(0) = 1$ is $y = 1$ and the solution satisfying $y(0) = a$ is $y = 1 + (a-1)e^{-x^2/2}$.

Example 11 *Solve $xy' - 2y = x^3 \sin x, (x > 0)$.*

We bring this linear first order equation to standard form by dividing by x. We get

$$y' + \frac{-2}{x}y = x^2 \sin x.$$

The integrating factor is

$$I(x) = e^{\int -2dx/x} = e^{-2\ln x} = 1/x^2.$$

After multiplying our DE in standard form by $1/x^2$ and simplifying, we get

$$\frac{d}{dx}(y/x^2) = \sin x$$

from which $y/x^2 = -\cos x + C$ and $y = -x^2 \cos x + Cx^2$. Note that the later are solutions to the DE $xy' - 2y = x^3 \sin x$ and that they all satisfy the initial condition $y(0) = 0$. This non-uniqueness is due to the fact that $x = 0$ is a singular point of the DE.

Definition 4 *A **Bernoulli Differential Equation** is a differential equation of the form*

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where n is an integer.

For $n = 1$, this equation reduces to a separable linear differential equation.

Theorem 3 *Suppose $n \neq 1$. Then the transformation $u = y^{1-n}$ reduces the equation to a first-order linear differential equation in u.*

Example 12 *Solve $y' + y = xy^3$.*

Solution: This is a Bernoulli DE with $n = 3$.
1) Let $u = y^{1-n} = y^{-2}$. Then $u' = (1-n)y^{-n}y' = -2y^{-3}y', \Rightarrow y^{-3}y' = -\frac{1}{2}u'$.
2) Divide the original equation by y^3 we have

$$y^{-3}y' + y^{-2} = x, \Rightarrow -\frac{1}{2}u' + u = x, \Rightarrow$$
$$u' - 2u = -2x. \qquad (1)$$

3) Now we have a linear DE. The integrating factor is:

$$I(x) = e^{\int -2dx} = e^{-2x}.$$

4) Multiply (1) by $I(x)$:

$$(e^{-2x}u)' = -2xe^{-2x}.$$

5) Integrating this we have

$$e^{-2x}u = \int -2xe^{-2x}dx.$$

By integration-by-parts,

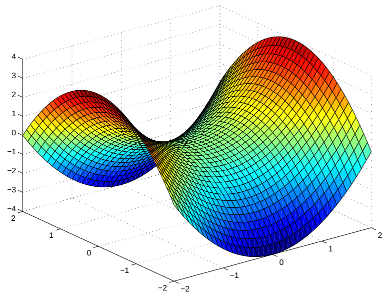
$$e^{-2x}u = xe^{-2x} + \frac{1}{2}e^{-2x} + C, \Rightarrow u = x + \frac{1}{2} + Ce^{2x}, y^{-2} = x + \frac{1}{2} + Ce^{2x}.$$

2.4 Functions of two variables

Definition: A function of two variables $z = f(x, y)$ is a relation which maps each point (x,y) in a set D in the xy-plane to a unique number z. The set D is called the domain of the function, which is often denoted D(f).

$\{(x, y, z) : (x, y) \in D, z = f(x, y)\}$ is called the graph of f (which is a surface).

Example. Sketch the graph of the function $z = x^2 - y^2$ (Hyperbolic Paraboloid):



Example. Classify the following quadric surfaces: $x^2 + 2z^2 - 6x - y + 10 = 0$.

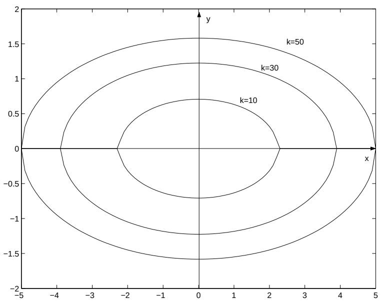
Solution: We change it to $\frac{(x-3)^2}{\sqrt{2}^2} + \frac{z^2}{1^2} = \frac{y-1}{2}$. This is an elliptic paraboloid with vertex (3, 1, 0), centered with the line x = 3, y = 1.

Example. Find the domain of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.

Solution: $4 - x^2 - y^2 \geq 0$, i.e., $x^2 + y^2 \leq 4$. Thus $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

Level curves (contour maps) of $f(x, y)$: $f(x, y) = k$ for different k.

Example. Let $f(x, y) = 2x^2 + 20y^2$. Sketch three level curves for $k = 10, 30, 50$.



Partial derivatives

- Partial derivatives of $z = f(x, y)$:

$$z_x = \frac{\partial z}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y) := D_x f := \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

which is the derivative of f with respect to x;

$$z_y = \frac{\partial z}{\partial y} := \frac{\partial f}{\partial y} := f_y(x, y) := D_y f := \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

which is the derivative of f with respect to y.

- Methods:
  1. To find f_x: regard y as a constant, and differentiate $f(x, y)$ with respect to x;
  2. To find f_y: regard x as a constant, and differentiate $f(x, y)$ with respect to y.
- Meaning: f_x means the rate of change of f with respect to x when y is fixed.

Example. Let $f(x, y) = e^{xy} + \frac{x}{y}$. Calculate $f_x(0, 1)$, $f_y(0, 1)$.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, \quad f_x(0, 1) = 2.$$
$$f_y = xe^{xy} - \frac{x}{y^2}, \quad f_y(0, 1) = 0.$$

Higher derivatives:

f_{xx}, $\frac{\partial^3 f}{\partial z \partial y \partial x} = f_{xyz}, \cdots$

Example. Let $f(x, y) = e^{xy} + \frac{x}{y}$. Calculate f_{xx}, f_{xy}, f_{yy}.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, f_y = xe^{xy} - \frac{x}{y^2}.$$
$$f_{xx} = y^2 e^{xy}, \quad f_{xy} = e^{xy} + xye^{xy} - \frac{1}{y^2}, \quad f_{yy} = x^2 e^{xy} + \frac{2x}{y^3}.$$

The Chain Rule

1. If $z = f(x, y)$, $x = g(t)$, $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2. $\frac{dz}{dt}$ means rate of change of z with respect to t along the path $x = g(t)$, $y = h(t)$, $t \in D$.

Example. Suppose $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$. Given the data

$$g(1) = 1,\ g'(1) = 2,$$
$$h(1) = 2, h'(1) = 3,$$
$$f_x(1,2) = -1,\ f_y(1,2) = 2.$$

Find $\frac{dz}{dt}$ when $t = 1$.

Solution. $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = f_x g' + f_y h'$$

$$t = 1 \Rightarrow (x, y) = (g(1), h(1)) = (1, 2)$$

$$\frac{dz}{dt}|_{t=1} = f_x(1,2)g'(1) + f_y(1,2)h'(1)$$
$$= (-1)(2) + (2)(3) = 4$$

Example. Consider the following function

$$z = x^2y + e^x \cos y, \quad x = t^3 \sin t,\ y = t^2.$$

Calculate $\frac{dz}{dt}$ by using Chain Rule.

2.5 Exact equations

By a region of the xy-plane we mean a connected open subset of the plane.

Definition 5 *The differential equation*

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

is said to be **exact** *on a region R if there is a function $f(x, y)$ defined on R such that*

$$df(x,y) = P(x,y)dx + Q(x,y)dy,$$

which is also called **exam differential**. *The function $f(x, y)$ is called potential function.*

Equivalently,
$$\frac{\partial f}{\partial x} = P(x,y); \quad \frac{\partial f}{\partial y} = Q(x,y)$$

Condition: The equation is exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{2}$$

The exact equations are solvable. In fact, if $f(x, y)$ is a potential function, then the solution of the exact equation is

$$f(x, y) = C.$$

Example 13 *Solve* $2x^2y\frac{dy}{dx} + 2xy^2 + 1 = 0.$

Solution: Here $P = 2xy^2 + 1$, $Q = 2x^2y$. The equation is exact since

$$\frac{\partial P}{\partial y} = 4xy = \frac{\partial Q}{\partial x}.$$

To find f we have to solve the partial differential equations

$$\frac{\partial f}{\partial x} = 2xy^2 + 1, \quad \frac{\partial f}{\partial y} = 2x^2y.$$

If we integrate the first equation with respect to x holding y fixed, we get

$$f(x, y) = x^2y^2 + x + \phi(y).$$

Differentiating this equation with respect to y gives

$$\frac{\partial f}{\partial y} = 2x^2y + \phi'(y) = 2x^2y$$

using the second equation. Hence $\phi'(y) = 0$ and $\phi(y)$ is a constant function. The solution is $x^2y^2 + x = C$.

Example 14 *Solve*

$$y - x + (x + y)\frac{dy}{dx} = 0.$$

This is an exact equation. The solution in implicit form is $x(y - x) + y(x + y) = C$, i.e., $y^2 + 2xy - x^2 = C$.

Integrating Factors.

If the differential equation $P + Qy' = 0$ is not exact, it can sometimes be made exact by multiplying it by a continuously differentiable function $I(x, y)$. Such a function is called an *integrating factor*. An integrating factor I satisfies the PDE:

$$\frac{\partial(IP)}{\partial y} = \frac{\partial(IQ)}{\partial x}$$

which can be written in the form

$$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) I = Q\frac{\partial I}{\partial x} - P\frac{\partial I}{\partial y}.$$

- Case 1: <u>I is a function of x only.</u> This happens if and only if

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$$

  is a function of x only, in which case

$$\frac{I'}{I} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}.$$

  We have

$$I(x) = \exp\left(\int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} dx\right).$$

- Case 2: <u>I is a function of y only.</u> This happens if and only if

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P}$$

  is a function of y only. We have

$$I(y) = \exp\left(\int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P} dy\right).$$

Example 15 *Solve* $2x^2 + y + (x^2y - x)y' = 0$.

Solution: Here

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{2 - 2xy}{x^2y - x} = \frac{-2}{x}$$

so that there is an integrating factor I which is a function of x only which satisfies $I'/I = -2/x$. Hence $I = 1/x^2$ is an integrating factor and $2 + y/x^2 + (y - 1/x)y' = 0$ is an exact equation whose general solution is $2x - y/x + y^2/2 = C$ or $2x^2 - y + xy^2/2 = Cx$.

Example 16 *Solve* $y + (2x - ye^y)y' = 0$.

Solution: Here

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P} = \frac{1}{y}$$

so that there is an integrating factor which is a function of y only which satisfies $I'/I = 1/y$. Hence y is an integrating factor and $y^2 + (2xy - y^2e^y)y' = 0$ is an exact equation with general solution $xy^2 + (-y^2 + 2y - 2)e^y = C$.

Remark. The solutions of the exact DE obtained by multiplying by the integrating factor may have solutions which are not solutions of the original DE. This is due to the fact that I may be zero and one will have to possibly exclude those solutions where I vanishes.

Chapter 3: Second-order Equations

3.1 Basic Definitions

A second order differential equation is an equation involving the unknown function y, its derivatives y' and y'', and the variable x. Linear second order DE:

$$a(x)y'' + b(x)y' + c(x)y = g(x).$$

Definition 6 *The functions* y_1, ..., y_n *are linearly independent on an interval* I *if*

$$c_1y_1 + \cdots c_ny_n = 0 \Rightarrow c_1 = \cdots = c_n = 0.$$

3.2 Linear Homogeneous Equations

Consider a linear, second-order, homogeneous equation in standard form,

$$y'' + p(x)y' + q(x)y' = 0. \tag{3}$$

Theorem 4 *If y_1 and y_2 are two linearly independent solutions, then the complementary function $y_c = c_1y_1 + c_2y_2$ is the general solution.*

Reduction of order

Suppose that one solution y_1 is known. Then a second, independent solution y_2 is obtained by letting $y_2(x) = u(x)y_1(x)$, $u(x)$ to be determined.

$$y_2 = uy_1 \to y_2' = u'y_1 + uy_1', \quad y'' = u''y_1 + 2u'y_1' + uy_1'',$$

so y_2 is a solution if and only if

$$[u''y_1 + 2u'y_1' + uy_1''] + p(x)[u'y_1 + uy_1'] + q(x)[uy_1] = 0,$$

i.e.,

$$u[y_1'' + p(x)y_1' + q(x)y_1] + u''y_1 + 2u'y_1' + p(x)u'y_1 = 0.$$

Since y_1 is a solution, $y_1'' + p(x)y_1' + q(x)y_1 = 0$. Thus, y_2 is a solution if and only if

$$u''y_1 + u'[2y_1' + p(x)y_1] = 0,$$

i.e.,

$$\frac{u''}{u'} = -\frac{2y_1' + p(x)y_1}{y_1} = -2\frac{y_1'}{y_1} - p(x).$$

Integration with respect to x then gives

$$\ln|u'| = -2\ln|y_1| - \int p(x)\,dx.$$

Taking the exponential of both sides and using the fact that $e^{-2\ln|y_1|} = e^{\ln|y_1|^{-2}} = \dfrac{1}{y_1^2}$, we obtain

$$|u'| = \frac{1}{y_1^2}e^{-\int p(x)\,dx}, \quad \text{or} \quad u' = \pm\frac{1}{y_1^2}e^{-\int p(x)\,dx}.$$

Taking the plus sign and integrating once more, we obtain

$$u(x) = \int \frac{1}{y_1^2}e^{-\int p(x)\,dx}\,dx.$$

Theorem 5 *If y_1 is a solution of (3), then the second linearly independent solution is:*

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\,dx.$$

Example 17 *Given that $y_1 = x^2$ is a solution of*

$$x^2 y'' - 3xy' + 4y = 0, \; x > 0,$$

find the general solution.

Solution:

$$u(x) = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\,dx = \int \frac{1}{x^4} e^{-\int -3/x\,dx}\,dx = \int \frac{x^3}{x^4} dx = \ln x.$$

Thus $y_2(x) = y_1(x) \ln x = x^2 \ln x$. The general solution is

$$y(x) = C_1 x^2 + C_2 x^2 \ln x.$$

3.2.1 Homogeneous linear equations with constant coefficients

Second-order homogeneous linear equation with constant coefficients

$$ay'' + by' + cy = 0.$$

Let r_1 and r_2 be the two solutions of the Auxiliary equation

$$ar^2 + br + c = 0.$$

- $r_1 \neq r_2$: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
- $r_1 = r_2$: $y = (C_1 + C_2 x) e^{rx}$.
- $r = \alpha + i\beta$: $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.

Example 18 *Find the general solution:*

1. $2y'' - 7y' - 4y = 0$,
2. $y'' + 6y' + 9y = 0$,
3. $y'' + 4xy' + 8y = 0$.

Example 19 *Solve the IVP:* $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

Solution: $y = e^{-x/2}(-\cos 2x + \frac{3}{4}\sin 2x)$.

3.2.2 Second-order Cauchy-Euler equation:

An equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + ... + a_1 x y' + a_0 y = 0,$$

is called a *Cauchy-Euler* equation.

Second-order Cauchy-Euler equation:

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0,$$

with a_2, a_1 and a_0 constants. In standard form, the equation is

$$y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0.$$

Let $y = x^r$. Find m so that y is a solution of the equation.

$$y' = r x^{r-1},\ y'' = r\,(r-1)\,x^{r-2}.$$

Substitute them into equation we get $x^r\,(a_2 r\,(r-1) + a_1 r + a_0) = 0$. Thus we get the auxiliary equation

$$a_2 r\,(r-1) + a_1 r + a_0 = 0, \quad or, \quad a_2 r^2 + (a_1 - a_2)\,r + a_0 = 0.$$

If $r_1 \neq r_2$ are real, then $y_1 = |x|^{r_1}$ and $y_2 = |x|^{r_2}$.

If $r_1 = r_2$ (real), then $y_1 = |x|^{r_1}$ and $y_2 = x^{r_1} \ln |x|$.

If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $y_1 = |x|^{\alpha} \cos(\beta \ln |x|)$ and $y_2 = |x|^{\alpha} \sin(\beta \ln |x|)$.

Proof. Since $p(x) = \dfrac{A}{x}$ and $q(x) = \dfrac{B}{x^2}$ are undefined at $x = 0$, the solution may be undefined at $x = 0$. Thus, we assume that $x \neq 0$. A Cauchy-Euler equation can be transformed into a constant-coefficient equation as follows:

For $x > 0$, let $x = e^t$ and $y(x) = z(t)$. Then $t = \ln(x)$ and, by the chain rule,

$$\frac{dy}{dx} = \frac{dz}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dz}{dt}, \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dz}{dt} + \frac{1}{x}\frac{d^2z}{dt^2}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dz}{dt} + \frac{1}{x^2}\frac{d^2z}{dt^2},$$

and the equation $x^2 y'' + Axy' + By = 0$ becomes $\left[\dfrac{d^2z}{dt^2} - \dfrac{dz}{dt}\right] + A\dfrac{dz}{dt} + Bz = 0$, or

$$z'' + (A-1)z' + Bz = 0,$$

which has constant coefficients.

If $z_1(t)$ and $z_2(t)$ are two independent solutions of $z'' + (A-1)z' + Bz = 0$, then two independent solutions of $x^2y'' + Axy' + By = 0$ are given by

$$y_1(x) = z_1(\ln x) \text{ and } y_2(x) = z_2(\ln x).$$

Since solutions of a constant-coefficient equation are sought in the form $z = e^{rt}$ and $y(x) = z(t)$ with $t = \ln(x)$, $y(x) = e^{rt} = e^{r\ln(x)} = e^{\ln(x^r)} = x^r$. Thus, solutions of an Euler equation can be sought directly in the form $y = x^r$.

If $r_1 \neq r_2$ are real, then $z_1 = e^{r_1 t}$ and $z_2 = e^{r_2 t} \rightarrow y_1 = x^{r_1}$ and $y_2 = x^{r_2}$.

If $r_1 = r_2$ (real), then $z_1 = e^{r_1 t}$ and $z_2 = te^{r_1 t} \rightarrow y_1 = x^{r_1}$ and $y_2 = x^{r_1}\ln(x)$.

If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $z_1 = e^{\alpha t}\cos(\beta t)$ and $z_2 = e^{\alpha t}\sin(\beta t) \rightarrow y_1 = x^{\alpha}\cos[\beta\ln(x)]$ and $y_2 = x^{\alpha}\sin[\beta\ln(x)]$.

For $x < 0$, let $x = -e^t$ and $y(x) = z(t)$. Then $t = \ln(-x)$, and the same equation for $z(t)$ results. In either case, $t = \ln|x|$.

Since $|x| = \left\{ \begin{array}{rl} x, & x > 0 \\ -x, & x < 0 \end{array} \right\}$, replacing x by $|x|$ gives the solutions for any $x \neq 0$. Thus,

If $r_1 \neq r_2$ are real, then $y_1 = |x|^{r_1}$ and $y_2 = |x|^{r_2}$.

If $r_1 = r_2$ (real), then $y_1 = |x|^{r_1}$ and $y_2 = x^{r_1}\ln|x|$.

If $r_1, r_2 = \alpha \pm i\beta$ (complex), then $y_1 = |x|^{\alpha}\cos(\beta\ln|x|)$ and $y_2 = |x|^{\alpha}\sin(\beta\ln|x|)$.

Example 20 *Solve the following equations:*

1. $x^2y'' + 2xy' - 2y = 0,\ x > 0.$
   Solution: This is an Euler equation. The auxiliary equation is

$$r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow y = c_1x^{-2} + c_2x = \frac{c_1}{x^2} + c_2x.$$

2. $x^2y'' + 5xy' + 4y = 0,\ x > 0.$

Solution: This is an Euler equation. The auxiliary equation is

$$r^2 + 4r + 4 = 0 \ \Rightarrow\ (r+2)^2 = 0 \ \Rightarrow\ y = \frac{c_1}{x^2} + c_2\frac{\ln(x)}{x^2}.$$

3. $x^2y'' + 4xy' + 4y = 0,\ x > 0.$

Solution: This is an Euler equation. The auxiliary equation is

$$r^2 + 3r + 4 = 0 \ \Rightarrow\ r = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}i \ \Rightarrow$$

$$y = x^{-\frac{3}{2}}\left[c_1 \cos\left(\frac{\sqrt{7}}{2}\ln(x)\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}\ln(x)\right)\right].$$

4. $x^2y'' - 6y = x^3\ln(x),\ x > 0.$

Solution: This is an Euler equation. The auxiliary equation is

$$r^2 - r - 6 = 0 \ \Rightarrow\ (r-3)(r+2) = 0 \ \Rightarrow\ y_1 = x^3,\ y_2 = x^{-2},\ y_h = c_1x^3 + c_2x^{-2}.$$

$f(x) = \dfrac{x^3\ln(x)}{x^2} = x\ln(x),$

$W[y_1, y_2] = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -5,\ u_1 = -\displaystyle\int \frac{y_2 f}{W}\,dx = \frac{1}{5}\int \frac{\ln(x)}{x}\,dx = \frac{1}{10}[\ln(x)]^2,$

$u_2 = \displaystyle\int \frac{y_1 f}{W}\,dx = -\frac{1}{5}\int x^4\ln(x)\,dx = -\frac{1}{25}x^5\ln(x) + \frac{1}{25}\int x^4\,dx =$

$-\dfrac{1}{25}x^5\ln(x) + \dfrac{1}{125}x^5 \ \Rightarrow$

$y_p = u_1y_1 + u_2y_2 = \dfrac{1}{10}[\ln(x)]^2x^3 + \left[-\dfrac{1}{25}x^5\ln(x) + \dfrac{1}{125}x^5\right]x^{-2} =$

$\dfrac{1}{10}[\ln(x)]^2x^3 - \dfrac{1}{25}x^3\ln(x) + \dfrac{1}{125}x^3 \ \Rightarrow$

$y = y_p + y_h = \dfrac{1}{10}[\ln(x)]^2x^3 - \dfrac{1}{25}x^3\ln(x) + \dfrac{1}{125}x^3 + c_1x^3 + c_2x^{-2}.$

3.3 Linear Nonhomogeneous Equations

Consider a linear, nonhomogeneous equation in standard form,

$$y'' + p(x)y' + q(x)y = G(x). \tag{4}$$

The general solution is

$$y = y_p + y_c,$$

where y_p is a particular solution of (4), and $y_c = c_1y_1 + c_2y_2$ is the general solution of the associated homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0. \tag{5}$$

3.3.1 The method of undetermined coefficients

- $G(x) = P(x)$ a polynomial with degree n.

  1. If x^k is not a solution of (5) for any $k \le n$, then let $y_p(x)$ be a polynomial with degree n.
  2. If x^k is a solution of (5) for $k \le n$, then let $y_p(x) = xQ(x)$, where $Q(x)$ is a polynomial with degree n.
  3. If x^k and x^s are solutions of (5) for $k \le n$ and $s \le n+1$, then let $y_p(x) = x^2Q(x)$, where $Q(x)$ is a polynomial with degree n.

- $G(x) = e^{kx}$.

  1. If e^{kx} is not a solution of (5), then let $y_p(x) = Ae^{kx}$.
  2. If e^{kx} is a solution of (5), but xe^{kx} not, then let $y_p(x) = Axe^{kx}$.
  3. If both e^{kx} and xe^{kx} are solutions of (5), then let $y_p(x) = Ax^2e^{kx}$.

- $G(x) = a\cos kx + b\sin kx$.

  1. If $\cos kx$ and $\sin kx$ are not solutions of (5), then let $y_p(x) = A\cos kx + B\sin kx$.

2. If $\cos kx$ or $\sin kx$ are solutions of (5), then let $y_p(x) = x[A\cos kx + B\sin kx]$.

- $G(x) = e^{kx}P(x)(s\cos mx + t\sin mx)$.
  Let $y_p(x) = e^{kx}[Q(x)\cos mx + R(x)\sin mx]$, where $\deg Q = \deg R = \deg P$.

- $G(x)$ is a combination of $P(x), e^{kx}, \cos kx, \sin kx$. Then use the Principle of Superposition.

The Principle of Superposition: If $y_1(x)$ and $y_2(x)$ are solutions of $y'' + p(x)y' + q(x)y = G_1(x)$ and $y'' + p(x)y' + q(x)y = G_2(x)$ respectively, then $y_1(x) + y_2(x)$ is a solution of $y'' + p_1(x)y' + q_1(x)y = G_1(x) + G_2(x)$.

Example 21 *Solve* $y'' - 2y' - 3y = 2 + x$.

Example 22 *Solve* $y'' - 2y' = 2 + x$.

Example 23 *Solve* $y'' - 2y' - 3y = 2e^x$.

Example 24 *Solve* $y'' + 2y' - 3y = 2e^x$.

Example 25 *Solve* $y'' - 4y' + 4y = 2e^{2x}$.

Example 26 *Solve* $y'' - 2y' - 3y = 2\cos(3x)$.

Example 27 *Solve* $y'' + 4y = 2\sin(2x)$.

Example 28 *Solve* $y'' + 4y = 8x^2 + 10e^x$.

3.3.2 Variation of parameters

Let y_1 and y_2 be two, independent solutions of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

We seek a particular solution y_p of the nonhomogeneous equation in the form $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, u_1 and u_2 to be determined. Then

$$y_p = u_1y_1 + u_2y_2 \to y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2').$$

The requirement that y_p be a solution of the nonhomogeneous equation imposes one condition upon the functions u_1 and u_2. Since there are two functions to be determined, we may impose a second condition upon them and, thus, we require that $u_1'y_1 + u_2'y_2 = 0$. Then

$$y_p' = u_1y_1' + u_2y_2' \to y_p'' = u_1y_1'' + u_1'y_1' + u_2'y_2' + u_2y_2'',$$

and y_p is a solution if and only if

$$[u_1y_1'' + u_1'y_1' + u_2'y_2' + u_2y_2''] + p(x)[u_1y_1' + u_2y_2'] + q(x)[u_1y_1 + u_2y_2] = f(x),$$

i.e.,

$$u_1[y_1'' + p(x)y_1' + q(x)y_1] + u_2[y_2'' + p(x)y_2' + q(x)y_2] + u_1'y_1' + u_2'y_2' = f(x).$$

Since y_1 and y_2 are solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, the latter requirement reduces to $u_1'y_1' + u_2'y_2' = f(x)$. Combining with the first condition imposed, we obtain the system of equations

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x) \end{aligned}$$

for the unknown quantities u_1' and u_2'. The system may be expressed in matrix form as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix},$$

with the solution

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ f(x) \end{pmatrix} = \frac{1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f(x) \end{pmatrix}.$$

The quantity $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$ is denoted by $W(x) = W[y_1(x)\ y_2(x)]$ and called the **Wronskian** of the functions y_1 and y_2. Thus,

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f(x) \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} -y_2f(x) \\ y_1f(x) \end{pmatrix} \to$$

$$u_1' = \frac{-y_2f}{W} \text{ and } u_2' = \frac{y_1f}{W} \to u_1 = -\int \frac{y_2f}{W}\,dx \text{ and } u_2 = \int \frac{y_1f}{W}\,dx.$$

Theorem 6 *If y_1 and y_2 are two linearly independent solutions of the equation*

$$y'' + p(x)y' + q(x)y = 0,$$

then a particular solution of

$$y'' + p(x)y' + q(x)y = f(x)$$

is given by

$$y_p = u_1 y_1 + u_2 y_2, \quad W(x) = y_1 y_2' - y_1' y_2, u_1 = -\int \frac{y_2 f}{W}\, dx,\ u_2 = \int \frac{y_1 f}{W}\, dx,$$

$$u_1' y_1 + u_2' y_2 = 0.$$

Example 29 *Solve the equation*

$$y'' + y = \tan x, \quad 0 < x < \pi/2.$$

Solution: 1) The auxiliary equation is: $r^2 + 1 = 0$, So, the solution of $y'' + y = 0$ is: $y_c = c_1 \sin x + c_2 \cos x$.

2) Using variation of parameters, we seek a solution of the form

$$y_p = u_1 y_1 + u_2 y_2 = u_1 \sin x + u_2 \cos x.$$

Then

$$y_p' = (u_1' \sin x + u_2' \cos x) + (u_1 \cos x - u_2 \sin x).$$

3) Set

$$u_1' \sin x + u_2' \cos x. \tag{6}$$

4) Then

$$y_p' = u_1 \cos x - u_2 \sin x.$$

$$y_p'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x.$$

For y_p to be a solution, we must have

$$y_p'' + y_p = u_1' \cos x - u_2' \sin x = \tan x. \tag{7}$$

5) Solve (6) and (7) we get

$$u_1' = \sin x, \Rightarrow u_1 = -\cos x.$$

$$u_2' = \cos x - \sec x, \Rightarrow u_2 = \sin x - \ln|\sec x + \tan x|.$$

Thus

$$y_p(x) = -\cos x \ln|\sec x + \tan x|.$$

3.4 Equations Reducible to the First Order Equations

Case 1: A second-order equation (linear or nonlinear) in which the dependent variable (e.g., y) does not appear explicitly can be reduced to a first-order equation by letting $u(x) = y'$.

Case 2: A second-order equation (linear or nonlinear) in which the independent variable (e.g., x) does not appear explicitly can be transformed into a first-order equation for y' as a function of y. Thus, given the second-order equation

$$f(y, y', y'') = 0,$$

let $y'(x) = u(y)$, then, by the chain rule, $y''(x) = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$, and the equation becomes $f(y, u, uu') = 0$, or $g(y, u, u') = 0$, which is of the first order.

For example, given $y'' + (y')^2\left(y + \frac{1}{y}\right) = 0$, in which x does not appear, let $y'(x) = u(y)$ to obtain the first-order equation $uu' + u^2\left(y + \frac{1}{y}\right) = 0$. Once u is determined by solving the first-order equation, y is obtained by solving the first-order equation $y' = u(y)$.

Example 30 *Solve* $y'' + 2x(y')^2 = 0$.

Solution: Since y does not appear explicitly in the equation, let $z = y'$. Then $y'' = z'$, and the equation becomes $z' + 2xz^2 = 0$, which is first-order and separable. Thus,
$z^{-2}z' = -2x \Rightarrow -z^{-1} = -x^2 + c \Rightarrow z = \frac{1}{x^2 - c} \Rightarrow y' = \frac{1}{x^2 - c} \Rightarrow y = \int \frac{dx}{x^2 - c}$.
If $c = 0$, then $y = -\frac{1}{x} + k$. If $c = -a^2 < 0$, then $y = \int \frac{dx}{x^2 + a^2}$.
$x = a\tan(t) \Rightarrow \frac{dx}{dt} = a\sec^2(t)$ and $x^2 + a^2 = a^2[\tan^2(t) + 1] = a^2\sec^2(t)$, so the integral transforms into $y = \frac{1}{a}\int dt = \frac{1}{a}t + k_1 = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + k_1$.
If $c = a^2 > 0$, then $y = \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \frac{1}{x - a} - \frac{1}{x + a}dx$ by partial fractions. Thus, $y = \frac{1}{2a}\left[\ln|x - a| - \ln|x + a|\right] + k_2 = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + k_2$.

Example 31 *Solve* $yy'' + (y')^2 = 0$.

Solution: Since x does not appear explicitly in the equation, put $y'(x) = u(y)$. Then $y'' = \frac{du}{dy}\frac{dy}{dx} = uu'$, and the equation becomes $yuu' + u^2 = 0$. Thus, either

$$u = 0, \Rightarrow y' = 0, \Rightarrow y = const;$$

or $u^{-1}u' = -y^{-1} \Rightarrow \ln|u| = -\ln|y| + c \Rightarrow u = \frac{k}{y} \Rightarrow yy' = k \Rightarrow \frac{1}{2}y^2 = kx + k_1 \Rightarrow$ $y = \pm\sqrt{k_2 x + k_3}$.

Chapter 4: Higher-order Linear Equations

4.1 Homogeneous Equations

4.1.1 Higher-order homogeneous linear equation with constant coefficients

$$a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_1 y' + a_0 y = 0.$$

The auxiliary (indicial) equation is

$$a_k r^k + a_{k-1} r^{k-1} + \cdots + a_1 r + a_0 = 0.$$

- If the auxiliary equation has k different roots r_i, then the solution is

$$y = c_1 e^{r_1 x} + \cdots + c_k e^{r_k x}.$$

- If the auxiliary equation has a root a with multiplicity m, then one solution is

$$y = e^{ax}(c_1 + c_2 x + \cdots + c_m x^{m-1}).$$

- If the auxiliary equation has a root $a + bi$ with multiplicity m, then one solution is

$$y = e^{ax}\cos(bx)[c_1 + c_2 x + \cdots + c_m x^{m-1}] + e^{ax}\sin(bx)[d_1 + d_2 x + \cdots + d_m x^{m-1}].$$

Example 32 *Solve the DE:* $y''' - 2y'' - y' + 2y = 0.$

Solution: $y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$.

Example 33 *Solve the DE:* $y''' + 3y'' - 4y = 0.$

Solution: $y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$.

Example 34 *Solve the DE:* $y^{(4)} + 2y'' + y = 0$.

Solution: $y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$.

Example 35 *Solve the DE:* $y^{(4)} - 12y^{(3)} + 52y'' - 156y' + 169y = 0$.

Solution: $y = c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x + c_3 x e^{3x} \cos 2x + c_4 x e^{3x} \sin 2x$.

Chapter 5: Linear Systems

5.1. Homogeneous systems

A linear system of differential equations has the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t),$$

$$\cdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t).$$

Letting $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, and $F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, we obtain the matrix differential equation $X' = AX + F$.

A linear system of homogeneous differential equations has the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n,$$

$$\cdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n.$$

For example, a system of two, linear, homogeneous equations with constant coefficients has the form

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy, \end{aligned}$$

where a, b, c and d are constants, and x and y are functions of t. The system may be expressed in matrix form as

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Letting $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we obtain the matrix differential equation $\mathbf{x}' = A\mathbf{x}$.

Superposition principle. Let X_i (i=1,...,n) be a set of solutions of the homogeneous system

$$\mathbf{X}' = A\mathbf{X}, \tag{8}$$

where $\mathbf{X}$ is a vector with n-entries of variables. Then the linear combination $c_1X_1 + ... + c_nX_n$ is also a solution.

Linear dependence/independence. If there exist n constants c_1,..., c_n, not all zero, such that $c_1X_1 + ... + c_nX_n = 0$ for all t in an interval, then X_i (i=1,...,n) are linearly dependent. Otherwise, independent.

Criterion for Linearly Independent Solutions. If the Wronskian $W(X_1, ..., X_n) = \det(X_1, ..., X_n) \neq 0$ for all t in an interval I, then X_i (i=1,...,n) are linearly independent on I.

Fundamental Set of Solutions. Any n linearly independent set of solutions of (8) is called a fundamental set of solutions of (8). The linear combination of any fundamental set of solutions of (8) is the general solution of (8).

Example 36 *Two large tanks, each holding* 24 *liters of brine, are interconnected by two pipes. Fresh water flows into tank A a the rate of* 6 *L/min, and fluid is drained out tank B at the same rate. Also,* 8 *L/min of fluid are pumped from tank A to tank B and* 2 *L/min from tank B to tank A. The solutions in each tank are well stirred sot that they are homogeneous. If, initially, tank A contains* 5 *in solution and Tank B contains* 2 *kg, find the mass of salt in the tanks at any time t.*

To solve this problem, let $x(t)$ and $y(t)$ be the mass of salt in tanks A and B respectively. The variables x, y satisfy the system of the first order DE

$$x' = \frac{dx}{dt} = \frac{-1}{3}x + \frac{1}{12}y,$$
$$y' = \frac{dy}{dt} = \frac{1}{3}x - \frac{1}{3}y.$$

The first equation gives $y = 12x' + 4x$. Substituting this in the second equation and simplifying, we get

$$x'' + \frac{2}{3}x' + \frac{1}{12}x = 0.$$

The general solution of this DE is

$$x = c_1 e^{-t/2} + c_2 e^{-t/6}.$$

This gives $y = -2c_1 e^{-t/2} + 2c_2 e^{-t/6}$. Thus the general solution of the system is

$$\begin{aligned} x &= c_1 e^{-t/2} + c_2 e^{-t/6}, \\ y &= -2c_1 e^{-t/2} + 2c_2 e^{-t/6}. \end{aligned}$$

These equations can be written in matrix form as

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t/2} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-t/6} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Using the initial condition $x(0) = 5$, $y(0) = 2$, we find $c_1 = 2$, $c_2 = 3$. Geometrically, these equations are the parametric equations of a curve (trajectory of the DE) in the xy-plane (phase plane of the DE). As $t \to \infty$ we have $(x(t), y(t)) \to (0, 0)$. The constant solution $x(t) = y(t) = 0$ is called an **equilibrium solution** of our system. This solution is said to be **asymptotically stable** if the general solution converges to it as $t \to \infty$. A system is called **stable** if the trajectories are all bounded as $t \to \infty$.

Here we consider homogeneous Linear system

$$\frac{dX}{dt} = AX.$$

Analogous to second-order, linear, homogeneous equations, we seek solutions in the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$, where λ is a constant and $\mathbf{v}$ is a vector to be determined.

$$\mathbf{x} = e^{\lambda t}\mathbf{v} \to \mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}, \text{ so } \mathbf{x}' = A\mathbf{x} \iff \lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v} \iff A\mathbf{v} = \lambda\mathbf{v},$$

i.e., λ is an eigenvalue of A with the corresponding eigenvector $\mathbf{v}$.

To find all eigenvalues, we solve the following characteristic equation

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix, and solve for λ. For each eigenvalue λ, the corresponding eigenvectors are the nonzero solutions $\mathbf{v}$ of the matrix equation $(A - \lambda I)\mathbf{v} = 0$.

(2×2) Homogeneous Linear systems

We now describe the solution of the system $\frac{dX}{dt} = AX$ for an arbitrary 2×2 matrix A. There are three main cases depending on whether the discriminant

$$\Delta = \operatorname{tr}(A)^2 - 4\det(A)$$

of the characteristic polynomial of A is > 0, < 0, $= 0$.

Case 1: If the 2×2 matrix A has two distinct real eigenvalues λ_1, λ_2. Then the matrix A has two independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, corresponding to the eigenvalues λ_1 and λ_2 respectively (distinct or not), then $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2$ are two independent solutions of the matrix equation, with the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, where c_1 and c_2 are arbitrary constants.

In terms of the original variables, if $\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$ and $\begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$, then

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \to \left\{ \begin{array}{rcl} x &=& c_1 e^{\lambda_1 t} v_{11} + c_2 e^{\lambda_2 t} v_{12} \\ y &=& c_1 e^{\lambda_1 t} v_{21} + c_2 e^{\lambda_2 t} v_{22} \end{array} \right\}.$$

Example 37 *Find the general solution:* $\left\{ \begin{array}{rcl} x' &=& 2x - y \\ y' &=& 3x - 2y \end{array} \right.$.

Solution: In matrix form, the system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \;\Rightarrow\; \lambda_1 = 1,\; \lambda_2 = -1.$

For $\lambda_1 = 1$, $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0} \;\Rightarrow\; -a + b = 0,\; a = 1 \;\Rightarrow\; \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -1$, $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0} \;\Rightarrow\; -3a + b = 0,\; a = 1 \;\Rightarrow\; \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Thus, $\mathbf{x}_1(t) = e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are independent solutions, and the general solution is $\mathbf{x}(t) = c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, i.e., $\begin{array}{l} x(t) = c_1 e^{t} + c_2 e^{-t} \\ y(t) = c_1 e^{t} + 3c_2 e^{-t} \end{array}$.

Case 2: If the 2×2 matrix A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$, and the matrix A has two independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then it is same as Case 1:

$$\vec{x}_1 = e^{\lambda t}\vec{v}_1, \quad \vec{x}_2 = e^{\lambda t}\vec{v}_2.$$

Example 38 *Find the general solution:* $\begin{cases} x' &= 3x \\ y' &= 3y \end{cases}$.

Solution: In matrix form, the system is $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.
Two eigenvalues are $\lambda_1 = \lambda_2 = 3$.
For $\lambda = 3$, $\Rightarrow$ $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Case 3: If the 2×2 matrix A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ with only one independent eigenvector $\mathbf{K}$, then one solution is given by $\mathbf{x}_1 = e^{\lambda t}\mathbf{K}$. A second linearly independent solution is given by

$$\mathbf{x}_2 = te^{\lambda t}K + e^{\lambda t}P,$$

where P satisfies

$$(A - \lambda I)P = K. \tag{9}$$

Example 39 *Find the general solution:* $\begin{cases} x' &= 3x - 18y \\ y' &= 2x - 9y \end{cases}$.

Solution: In matrix form, the system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$.
Two eigenvalues are $\lambda_1 = \lambda_2 = -3$.
For $\lambda_1 = -3$, $\Rightarrow$ $\mathbf{v}_1 = K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\Rightarrow$ $\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
To find a second linearly independent solution, we solve (9),

$$(A + 3I)P = K, \Rightarrow P = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}.$$

Thus, $\mathbf{x}_2(t) = te^{\lambda t}K + e^{\lambda t}P = te^{-3t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$, and the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Case 4: If the 2×2 matrix A has complex eigenvalues, $\lambda = a + bi$. Let $\mathbf{K}$ be the corresponding eigenvector. Then

$$X_1 = (B_1 \cos bt - B_2 \sin bt)e^{at}, X_2 = (B_2 \cos bt + B_1 \sin bt)e^{at}, \quad B_1 = Re\mathbf{K}, B_2 = Im\mathbf{K}.$$

In fact, $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2$ are two complex solutions. In this case the roots of the characteristic polynomial are complex numbers

$$r = \alpha \pm i\omega = \text{tr}(A)/2 \pm i\sqrt{\Delta/4}.$$

The corresponding eigenvectors of A are (complex) scalar multiples of

$$\begin{bmatrix} 1 \\ \sigma \pm i\tau \end{bmatrix}$$

where $\sigma = (\alpha - a)/b$, $\tau = \omega/b$. If X is a real solution we must have $X = V + \overline{V}$ with

$$V = \frac{1}{2}(c_1 + ic_2)e^{\alpha t}(\cos(\omega t) + i\sin(\omega t))\begin{bmatrix} 1 \\ \sigma + i\tau \end{bmatrix}.$$

It follows that

$$X = e^{\alpha t}(c_1 \cos(\omega t) - c_2 \sin(\omega t))\begin{bmatrix} 1 \\ \sigma \end{bmatrix} + e^{\alpha t}(c_1 \sin(\omega t) + c_2 \cos(\omega t))\begin{bmatrix} 0 \\ \tau \end{bmatrix}.$$

The trajectories are spirals if $\text{tr}(A) \neq 0$ and ellipses if $\text{tr}(A) = 0$. The system is asymptotically stable if $\text{tr}(A) < 0$ and unstable if $\text{tr}(A) > 0$.

Example 40 *Find the general solution:* $\begin{cases} x' = 6x - y \\ y' = 5x + 4y \end{cases}.$

Solution: In matrix form, the system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$.

Two eigenvalues are $\lambda_1 = 5 + 2i$, $\lambda_2 = 5 - 2i$.

The general solution is

$$X = e^{5t}(c_1 \cos(2t) - c_2 \sin(2t))\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{5t}(c_1 \sin(2t) + c_2 \cos(2t))\begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Chapter 6: Sequences and Series

6.1 Sequences

Sequence:

$$a_1, a_2, ..., a_n, ...$$

a_n is the nth term. If $\lim_{n\to\infty} a_n$ exists, then we say the sequence converges. Otherwise, we say the sequence diverges.

Example. The sequence $\left\{\frac{\ln n}{n}\right\}$ is convergent.

Solution:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Example. The sequence $\left\{\sqrt{n^2+3n-1} - \sqrt{n^2-1}\right\}$ converges to $\frac{3}{2}$.

Example. The sequence $\{\cos n\}$ is divergent; $\{\arctan(-n)\}$ converges to $-\frac{\pi}{2}$.

Example.

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r| < 1; \\ \infty, & |r| > 1. \end{cases}.$$

Thus the sequence $\left\{(\frac{1}{3})^n\right\}$ is convergent, but the sequence $\{(3)^n\}$ is divergent.

Property: If $a_n = f(n)$ and $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

Properties: Let $\{a_n\}$ and $\{b_n\}$ be convergent, $c, d \in \mathbb{R}$.

1. $\lim\limits_{n\to\infty}(ca_n + db_n) = c \lim\limits_{n\to\infty} a_n + d \lim\limits_{n\to\infty} b_n$.
2. $\lim\limits_{n\to\infty}(a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$.
3. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim\limits_{n\to\infty} b_n \neq 0$.
4. $\lim\limits_{n\to\infty} a_n^p = (\lim\limits_{n\to\infty} a_n)^p$, if $a_n \geq 0$ and $p > 0$.

5. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Squeeze Theorem: If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example. $\{\frac{\cos n}{n}\}$ converges 0.

Bounded above: $a_n \le M$ for all n.
Bounded below: $a_n \ge m$ for all n.
Bound and Convergence: A convergent sequence is bounded.
Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.
Example. Given the sequence $\{a_n\}$: $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$.

(a) Show that the sequence is increasing and bounded above.

Solution: (i) We will show that $a_n < 6$ by induction: Note that $a_1 < 6$. Assume that $a_n < 6$. Then $a_{n+1} = \frac{1}{2}(a_n + 6) < \frac{1}{2}(6 + 6) = 6$.

(ii) $a_{n+1} - a_n = 3 - \frac{1}{2}a_n > 3 - \frac{6}{2} = 0$. Thus the sequence is increasing. (b) Find the limit of the sequence.

Solution: By Monotonic Sequence Theorem, the sequence is convergent. Let the limit be L. Then by the recursive relation, $L = \frac{1}{2}(L + 6) \Rightarrow L = 6$.

Example. $a_n = \frac{n}{n^3+1}$ is decreasing, and $0 < a_n < 1$ for any n.

To check it, let $f(x) = \frac{x}{x^3+1}$. Then $f'(x) = \frac{1-2x^2}{(x^3+1)^2} < 0$ when $x \ge 1$. Hence $f(x)$ is decreasing when $x \ge 1$.

6.2 Series

A basic fact is

$$\lim_{n\to\infty} r^n = 0 \Leftrightarrow |r| < 1.$$

The sum $\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + \ldots$ is called infinite series.
Partial sum:

$$S_n = \sum_{j=1}^{n} a_j.$$

Then

$$\sum_{j=1}^{\infty} a_j = S \Leftrightarrow \lim_{n\to\infty} S_n = S.$$

- Geometric series: if $|r| < 1$ then $\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$.
- $\frac{k}{n(n+k)} = \frac{1}{n} - \frac{1}{n+k}$.
- Harmonic series $\sum_{j=1}^{\infty} \frac{1}{n}$ is divergent.
- Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{j=1}^{\infty} a_j$ is divergent.
- $\sum_{j=1}^{\infty}(ca_j + db_j) = c\sum_{j=1}^{\infty} a_j + d\sum_{j=1}^{\infty} b_j$.

Example. Determine if the series converges or diverges:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+2}}{3^{n+1}}$

(b) $\sum_{n=1}^{\infty} 3^{n+1}2^{-2n}$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{\ln n}$.

(d) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$.

Solution:(a)

$$\sum_{n=0}^{\infty} \frac{2^{2n+2}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{2^{2(n+1)}}{3^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^{n+1}.$$

It is a geometric series with ratio $r = 4/3 > 1$. Therefore it is divergent.

(b) We have

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n} = \sum_{n=1}^{\infty} 3 \cdot 3^n 4^{-n} = 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

This series is a geometric series with first term $a = 9/4$ and common ratio $r = 3/4$. Since $|r| < 1$, the series converges and we have

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n} = \frac{a}{1-r} = \frac{9/4}{1/4} = 9.$$

(c) Divergent by Divergence Test.

(d)

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots\right) = \frac{5}{12}.$$

6.2.1 The Integral Test

Integral test: If $f(x)$ is continuous, positive, decreasing, $f(j) = a_j$, then

$$\sum_{j=1}^{\infty} a_j \text{ is convergent} \Leftrightarrow \int_1^{\infty} f(x)dx \text{ is convergent.}$$

Example. Determine if the series converges or diverges $\sum_{n=3}^{\infty} \frac{3}{n(\ln n)^3}$

Sol: We use Integral Test. Let $f(x) = \frac{3}{x(\ln x)^3}$. Then $f(x) > 0$ for $x \geq 3$. Since

$$f'(x) = \frac{-3[(\ln x)^3 + 3(\ln x)^2]}{x^2(\ln x)^6} < 0,$$

$f(x)$ is decreasing. By substitution with $u = \ln x$, we have

$$\begin{aligned}\int_3^{\infty} \frac{3}{x(\ln x)^3}dx &= \lim_{b\to\infty} \int_3^{b} \frac{3}{x(\ln x)^3}dx \\ &= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} \frac{3}{u^3}du \\ &= \frac{3}{2}(\frac{1}{\ln 3})^2.\end{aligned}$$

Hence it is convergent.

Example. Show that the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges.

The $p-$series: $\sum_{j=1}^{\infty} \frac{1}{j^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example. For what value of p is the series

$$\sum_{n=1}^{\infty} n^{p-1}$$

convergent?

Solution:

$$\sum_{n=1}^{\infty} n^{p-1} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{-p+1}.$$

Hence, if $-p + 1 > 1$, i.e., $p < 0$, then the series is convergent.

6.2.2 The Comparison Tests

Comparison test: if $0 \le a_j \le b_j$, then

$$\sum_{j=1}^{\infty} b_j \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

$$\sum_{j=1}^{\infty} a_j \text{ is divergent} \Rightarrow \sum_{j=1}^{\infty} b_j \text{ is divergent.}$$

Example. Determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{n(n^3+5)^{1/3}}$.

Solution: By Comparison test,
$0 \le \sum_{n=1}^{\infty} \frac{1}{n(n^3+5)^{1/3}} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ (p-series, $p = 2 > 1$), so convergent.

The Limit Comparison test: Suppose that $a_j > 0$ and $b_j > 0$ for any j, and

$$\lim_{j\to\infty} \frac{a_j}{b_j} = c > 0.$$

Then either both $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are convergent or both diverge.

Example. Determine if the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{3n^3 + 2n + 1}{9n^3\sqrt{n} + 1}.$$

Solution. Note that

$$a_n = \frac{3n^3 + 2n + 1}{9n^3\sqrt{n} + 1} \sim \frac{3n^3}{9n^3\sqrt{n}} = \frac{1}{3\sqrt{n}}.$$

Let

$$b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}.$$

Then

$$\sum_{n=1}^{\infty} b_n$$

is divergent by p-series test with $p = 1/2$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3},$$

the given series is divergent by Limit Comparison Test.

Remainder estimate: Let $R_n = \sum_{j=n+1}^{\infty} a_j$. Then

$$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx.$$

Example. (1) Estimate the error if we use the sum of the first 10 terms to approximate

$$\sum_{n=1}^{\infty} \frac{3}{n^4}.$$

(2) Find the smallest n such that S_n is within 0.000001 of the sum.

Solution. (1) Let $f(x) = \frac{3}{x^4}$. Then

$$\int_{11}^{\infty} f(x)dx \le R_{10} \le \int_{10}^{\infty} f(x)dx, \Rightarrow \frac{1}{11^3} \le R_{10} \le \frac{1}{10^3}.$$

(2)

$$R_n \le \frac{1}{n^3}, \Rightarrow \frac{1}{n^3} \le 0.000001, \Rightarrow n \ge 100,$$

$$R_{99} \ge \int_{100}^{\infty} f(x)dx = 0.000001.$$

6.2.3 Alternating series

1. **Alternating series Test:** The alternating series $\sum_{j=1}^{\infty}(-1)^{j-1}b_j$ is convergent if

   1. $b_j > 0$
   2. b_j decreasing $(b_1 \ge b_2 \ge b_3 \ge \cdots)$
   3. $\lim_{j\to\infty} b_j = 0$.

For the alternating series, we can estimate the **remainder** by using the following inequality

$$|R_n| \le b_{n+1}.$$

Example. Test the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n^2+n+1}$ for convergence or divergence.

Solution: This is an alternating series. Let $b_n = \frac{n+1}{n^2+n+2}$. Then

1) $\lim_{n\to\infty} b_n = 0$;

2) Let $f(x) = \frac{x+1}{x^2+x+1}$ for $x \ge 1$, then

$$f'(x) = \frac{-x^2-2x}{(x^2+x+1)^2} < 0.$$

Therefore $f(x)$ decreases for $x \geq 1$. In particular,

$$f(n) > f(n+1)$$

for all positive integer n. Hence b_n decreases.

By the Alternating Series Test, the series converges.

6.2.4 Absolute and Conditional Convergence

1. Absolute and Conditional Convergence If the series $\sum_{j=1}^{\infty} |a_j|$ is convergent, then we say that $\sum_{j=1}^{\infty} a_j$ is absolutely convergent; If the series $\sum_{j=1}^{\infty} a_j$ is convergent, but $\sum_{j=1}^{\infty} |a_j|$ is divergent, then we say that $\sum_{j=1}^{\infty} a_j$ is conditionally convergent.

Property:

$$\sum_{j=1}^{\infty} |a_j| \text{ is convergent} \Rightarrow \sum_{j=1}^{\infty} a_j \text{ is convergent.}$$

Example. Determine whether the series is absolutely convergent: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n^2+n+1}$.

Solution:

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{n+2}{n^2+n+1}\right| = \sum_{n=1}^{\infty} \frac{n+2}{n^2+n+1}.$$

Note that

$$\frac{n+2}{n^2+n+1} \geq \frac{n}{3n^2} = \frac{1}{3n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{3n}$$

is divergent (the harmonic series), by Comparison Theorem, the original series is not absolutely convergent.

By Alternating Series Test, the series is convergent, therefore the series is conditionally convergent.

Example. Show that the series is absolutely convergent: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(5n)}{n^2+7n+3}$.

2. Ratio test: Consider the series $\sum_{j=1}^{\infty} a_j$, with $\lim_{j\to\infty} |\frac{a_{j+1}}{a_j}| = L$.

- If $L < 1$, then the series is absolutely convergent;

- If $L > 1$, then the series is divergent.
- If $L = 1$, then the test is inconclusive.

Example. Test the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ for convergence or divergence.

Solution: Let $a_n = \frac{n!}{n^n}$. Then

$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1.$$

By the Ratio Test, the series converges.

3. Root test: Consider the series $\sum_{n=1}^{\infty} a_n$, with $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

- If $L < 1$, then the series is absolutely convergent;
- If $L > 1$, then the series is divergent.
- If $L = 1$, then the test is inconclusive.

Example. Test the series $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ for convergence or divergence.

Solution: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$. By the Root Test, the series converges.

Example. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2\sqrt[n]{n}+1)^n}$ is convergence.

Solution: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2e^0+1} = \frac{1}{3} < 1$. By the Root Test, the series converges.

Chapter 7: Taylor Series

7.1 Power Series

Definition. The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called power series, a is called the center. There exists $R \geq 0$ such that the series is convergent in $|x-a| < R$ and divergent in $|x-a| > R$. R is called radius of convergence.

- Radius of convergence: Using Ratio Test to find R.

- Interval of convergence I: Symmetric to the center a, with two end points $a-R$ and $a+R$. The convergence or divergence at the two end points $x=a-R$ and $x=a+R$ should be checked.

Example. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n}$.

Sol: We use Ratio Test.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)\,5^{n+1}} / \frac{(-1)^n(x-1)^n}{n\,5^n}| = \lim_{n\to\infty} \frac{1}{5}\frac{n}{n+1}|x-1|$$

$$= \frac{1}{5}|x-1| = L.$$

When $L < 1$, we have $|x-1| < 5$. Hence

$$R = 5,$$

and $-4 < x < 6$.

When $x = -4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-4-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent (Harmonic series).

When $x = 6$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(6-1)^n}{n\,5^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

which is convergent by Alternating Series Test. Therefore,

$$I = (-4, 6].$$

Example. Find the radius and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{n^2}$$

Solution: Let

$$a_n = \frac{(2x-3)^n}{n^2}.$$

Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot |(2x-3)| = |2x-3|.$$

By Ratio Test, $|2x-3|<1$, i.e., $|x-1.5|<0.5$. Therefore $R=0.5$ and $1<x<2$.

When $x=1$,

$$\sum_{n=1}^{\infty}\frac{(2x-3)^n}{n^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

which is convergent by Alternating Series Test.

When $x=2$,

$$\sum_{n=1}^{\infty}\frac{(2x-3)^n}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is convergent by $p-$Series Test. Therefore,

$$I=[1,2].$$

Example. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^{3n}}{n\,8^n}$.

Sol: We use Ratio Test.

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim_{n\to\infty}\frac{1}{8}\frac{n}{n+1}|x-1|^3$$
$$=\frac{1}{8}|x-1|^3.$$

By Ratio Test, we have $|x-1|^3<8$, i.e., $|x-1|<2$. Hence

$$R=2,\quad and\quad -1<x<3.$$

When $x=-1$,

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^{3n}}{n\,8^n}=\sum_{n=1}^{\infty}\frac{1}{n},$$

which is divergent (Harmonic series).

When $x=3$,

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^{3n}}{n\,8^n}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n},$$

which is convergent by Alternating Series Test. Therefore,

$$I=(-1,3].$$

7.2 Representations of Functions by Power Series

- Basic result:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

- Term-by-term differentiation: If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

then

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

and f and f' have the same radius of convergence.

- Term-by-term integration: If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

then

$$\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1},$$

and f and $\int f dx$ have the same radius of convergence.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1.$

(ii) $\frac{1}{1+2x^2} = \frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty}(-2x^2)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n}, \quad |-2x^2| < 1.$

(iii) $\frac{x^3}{6x+3} = \frac{x^3}{3}\frac{1}{1+2x} = \frac{x^3}{3}\sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{3}(x)^{n+3}.$

To find the domain,

Method 1: This series is convergent when $|-2x| < 1$, which is $|x| < 0.5$. So $I = (-0.5, 0.5)$.

Method 2: Let $a_n = (-1)^n \frac{2^n}{3}(x)^{n+3}$. Then

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{2^{n+1}}{3}(x)^{n+4}}{(-1)^n\frac{2^n}{3}(x)^{n+3}}\right| = 2|x|.$$

By the Ratio Test, when $L < 1$, we have $2|x| < 1$, $|x| < 0.5$. Hence $R = 0.5$. When $x = \pm 0.5$, $(-1)^n \frac{2^n}{3}(x)^{n+3} \not\to 0$, the series diverge. So $I = (-0.5, 0.5)$.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $f(x) = \frac{1}{(1+x)^2}$.

$$\frac{1}{(1+x)^2} = \frac{d}{dx}(\frac{-1}{1+x}) = \frac{d}{dx}[(-1)\sum_{n=0}^{\infty}(-x)^n] = \sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}, \quad -1 < x < 1.$$

(ii) $f(x) = \frac{1}{(1-x)^3}$.

Example. Represent the following functions as power series and determine the domain of the series.

(i) $f(x) = \ln(1+x)$.

Solution: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$, $R = 1$. $I = (-1, 1]$.

(ii) $f(x) = \arctan x$.

Solution:

$$\arctan x = \int \frac{1}{1+x^2}dx = \int (\sum_{n=0}^{\infty}(-1)^n x^{2n})dx$$
$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad R = 1.$$

Note that $C = \arctan 0 = 0$, we have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad -1 \le x \le 1.$$

Remark. It's difficult to check the two end-points.

Example. (i) Represent $\int \frac{1}{1+x^3}dx$ as power series and use it to approximate $\int_0^{0.1} \frac{1}{1+x^3}dx$ correct to within 0.00001.

Example. Represent $\frac{3x-1}{x^2-1}$ as a power series.

Solution:

$$\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1} = \sum_{n=0}^{\infty}[2(-1)^n - 1]x^n.$$

Example. Represent $\frac{1}{x}$ as a power series, centered at 1.

Solution:

$$\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n, \quad |x-1| < 1.$$

Taylor and Maclaurin Series

- Taylor series for $f(x)$ at the center a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

- Maclaurin series for $f(x)$ = Taylor series for $f(x)$ at the center 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n;$$

- Taylor polynomial:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

$$f(x) \approx T_n(x).$$

- Taylor's inequality: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then on $|x-a| \leq d$,

$$|R_n(x)| = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

- Taylor Theorem: If $f(x) = T_n(x) + R_n(x)$, and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then $f(x)$ is equal to its Taylor series for $|x-a| < R$. To this end, the following result is useful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

- Series for composite functions;

- Multiplication and division of power series.

Maclaurin series for some special functions

Example. Maclaurin series for some special functions:

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...,\ R = \infty;$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...,\ R = \infty;$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ..., R = \infty;$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...,\ R = 1;$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ..., \quad R = 1.$

Series for composite functions

Example. Maclaurin series:

(i) e^{x^2}, $\sin(x^2)$.

(ii)

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + ...$$
$$= 1 + (x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...) + (\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)^2}{2!}) + (\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)^3}{3!}) + ...$$
$$= 1 + x + \frac{x^2}{2!} + 0x^3 + ...$$

Binomial series

If k is a real number and $|x| < 1$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n,$$

here

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}, \quad \binom{k}{0} = 1.$$

Application: Let $f(x) = (1+x)^k$, then

$$f^{(n)}(0) = \binom{k}{n} n! = k(k-1)\cdots(k-n+1).$$

Example. Maclauring series for $f(x) = \sqrt[3]{1+x}$.

Solution:

$$\sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{2}{3^2 2!}x^2 + \frac{2 \cdot 5}{3^3 3!}x^3 - \frac{2 \cdot 5 \cdot 8}{3^4 4!}x^4 + \cdots.$$

Example. Let $f(x) = \sqrt[5]{1+x^2}$. Evaluate $f^{(4)}(0)$.

Solution. Use the binomial series to find the Maclaurin series of $f(x)$.

$$\frac{f^{(n)}(0)}{n!} = \binom{k}{n}$$

Hence

$$f^{(n)}(0) = \binom{k}{n} n! = k(k-1)\cdots(k-n+1).$$

So $f^{(4)}(0) = -0.8064$.

Taylor series at other centers

Example. Find the Taylor series for $f(x) = \sin x$ at the center $x = \frac{\pi}{3}$.

Multiplication and division of Taylor series

Example. Evaluate

$$\lim_{x\to 0} \frac{e^x \sin x - x}{x^2}.$$

$$\begin{aligned} e^x \sin x &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(x - \frac{x^3}{3!} + ...\right) \\ &= x + x^2 + \frac{1}{3}x^3 + ... \end{aligned}$$

Therefore

$$\frac{e^x \sin x - x}{x^2} = 1 + \frac{1}{3}x + ...$$

Example. Let

$$f(x) = \int_0^x t^3 e^{3t} dt$$

(a) Find the Maclaurin series of the function f.

(b) Find the radius of the series in (a).

Solution: a)

$$t^3 e^{3t} = t^3 \sum_{n=0}^{\infty} \frac{(3t)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} t^{n+3}.$$

Hence

$$f(x) = \int_0^x \sum_{n=0}^{\infty} \frac{3^n}{n!} t^{n+3} dt$$

$$= \sum_{n=0}^{\infty} \int_0^x \frac{3^n}{n!} t^{n+3} dt = \sum_{n=0}^{\infty} \frac{3^n}{n!} \frac{t^{n+4}}{n+4} |_0^x$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!(n+4)} x^{n+4}$$

b)

$$|a_{n+1}/a_n| = \frac{3(n+4)}{(n+1)(n+5)} |x| \to 0$$

as $n \to \infty$. Therefore $R = \infty$.

Example. Find the first 5 non-zero terms in the Maclaurin series for $e^x \cos(3x)$.

Solution. Note that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Hence

$$\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + ...$$

$$= 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - ...$$

We have

$$e^x \cos(3x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$$

$$- \frac{9x^2}{2} - \frac{9x^3}{2} - \frac{9x^4}{4} - ...$$

$$+ \frac{27x^4}{8} + ...$$

$$= 1 + x + (\frac{1}{2} - \frac{9}{2})x^2 + (\frac{1}{6} - \frac{9}{2})x^3 + (\frac{1}{24} - \frac{9}{4} + \frac{27}{8})x^4 + ...$$

$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \frac{7}{6}x^4 + ...$$

Example. Find the first 3 non-zero terms in the Maclaurin series for $\tan x$.

Solution: $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$

Applications of Taylor series.

Example.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

Example. Calculate the sum of the following series, given that it is a known series evaluated at a certain value of x:

$$1 - \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} - \frac{1}{64 \cdot 6!} + \frac{1}{256 \cdot 8!} - \cdots$$

Sol:

$$1 - \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} - \frac{1}{64 \cdot 6!} + \frac{1}{256 \cdot 8!} - \cdots$$

$$= 1 - \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 - \frac{1}{6!}\left(\frac{1}{2}\right)^6 + \frac{1}{8!}\left(\frac{1}{2}\right)^8 - \cdots$$

$$= \cos\frac{1}{2}.$$

Taylor polynomials and approximations

Taylor polynomial of degree n approximating $f(x)$ for x at a:

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $n = 1$, we have the linear approximation.

Example. Taylor polynomials of $f(x) = \sin x$ at $x = 0$:

$$P_1(x) = f(a) + f'(a)(x-a) = f(0) + f'(0)(x-0) = \sin 0 + (\cos 0)x = x,$$

$$P_3(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3 = \sin 0+(\cos 0)x-\frac{\sin 0}{2!}x^2-\frac{\cos 0}{3!}x^3$$

$$= x - \frac{x^3}{6}.$$

Example. Let $f(x)$ be a function such that

$$f(1) = 0, f'(1) = \frac{1}{5}, f''(1) = \frac{1}{10}, f'''(1) = \frac{1}{25}.$$

Estimate $f(1.15)$ using the Taylor expansion with order 3.

$$f(1.15) \approx T_3(1.15) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(1.15-1)^n = 0.0311475$$

.

Error in Taylor polynomial approximation: Let $P_n(x)$ be the Taylor approximation of $f(x)$ at $x = a$, then Taylor's inequality (The Lagrange Error Bound): If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then on $|x-a| \leq d$,

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Example. Give a bound on the error E_3 to the function e^{2x} about 0 for $-1 \leq x \leq 1$.

Sol: Let $f(x) = e^{2x}$.

$$P_3(x) = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!}.$$

$$f^{(3+1)}(x) = f^{(4)}(x) = 2^4 e^{2x}, \Rightarrow |f^{(3+1)}(x)| \leq 16e^2, |x| \leq 1.$$

$$|E_3(x)| = |f(x) - P_3(x)| \leq \frac{|f^{(3+1)}(x)|}{(3+1)!}|x-0|^{3+1} \leq \frac{16e^2}{4!}1^{3+1} = 2e^2/3.$$

Chapter 8: Fourier Series

8.1 Fourier series of periodic functions

A function $f(x)$ is **piecewise continuous** in interval (a, b) if we have $a = t_0 < t_1 < \ldots < t_m = b$, such that f(x) is continuous in each interval (t_i, t_{i+1}) and the limits $\lim_{x \to t_i^-} f(x)$ and $\lim_{x \to t_i^+} f(x)$ exist for all $i = 0, 1, 2, \ldots, m$. In the following, we assume that both f and f' are piecewise continuous.

A function $f(x)$ is called $p - periodic$ if $p > 0$ is the smallest number such that $f(x + p) = f(x)$ for any x. The number p is called the period. For example, $\cos kx$ and $\sin kx$ are $\frac{2\pi}{k}$-periodic.

Definition. Let $f(x)$ be $2L$-periodic function. Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right\}. \tag{10}$$

This series is called the (full) Fourier series for $f(x)$. The coefficients a_n $(n \geq 0)$ are called the Fourier cosine coefficients, and the coefficients b_n $(n \geq 1)$ are called the Fourier sine coefficients.

Remark. The "=" occurs at every $x \in [-L, L]$ where f is continuous. If we omit the condition where f is continuous at x, then we may write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})).$$

Theorem. The Fourier coefficients can be calculated as follows:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx \qquad n = 0, 1, 2, \ldots. \tag{11}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx \qquad n = 1, 2, 3, \ldots. \tag{12}$$

Proof. The coefficient a_0 is the simplest to find: integrating (14) from $-L$ to L,

$$\begin{aligned}\int_{-L}^{L} f(x)\,dx &= \int_{-L}^{L} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty}\left\{a_n \int_{-L}^{L} \cos(\frac{n\pi x}{L})\,dx + a_n \int_{-L}^{L} \sin(\frac{n\pi x}{L})\,dx\right\} \\ &= \int_{-L}^{L} \frac{a_0}{2}\,dx\end{aligned}$$

The series on the right vanishes, and we find that

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

We do the same thing to compute, say, b_m, except that first we multiply (14) through by $\sin(\frac{m\pi x}{L})$. We get

$$\begin{aligned}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx &= \int_{-L}^{L} \frac{a_0}{2}\sin(\frac{m\pi x}{L})\,dx + \\ &\sum_{n=1}^{\infty}\left\{a_n \int_{-L}^{L} \cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx + b_n \int_{-L}^{L} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx\right\}.\end{aligned}$$

What is important to notice is that *all* of the integrals on the right side vanish, except for the one multiplying b_m. The equation for b_m becomes

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx \qquad m = 1, 2, 3, \ldots.$$

Likewise we can get the formula for a_m.

Formulas (11) and (12) allow us to compute the Fourier coefficients of f.

Example 41 *Let*

$$f(x) = \begin{cases} 0, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in (0, \pi). \end{cases},$$

and let $f(x)$ be 2π-periodic. We will compute the Fourier coefficients for this function.

Solution:

$$\begin{aligned}a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} 1dx\right) = 1, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(nx)dx + \int_{0}^{\pi} 1\cos(nx)dx\right) = \frac{1}{n\pi}\sin(nx)|_0^{\pi} = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(nx)dx + \int_{0}^{\pi} 1\sin(nx)dx\right) = -\frac{1}{n\pi}\cos(nx)|_0^{\pi} \\ &= \frac{1}{n\pi}(1-(-1)^n) = \begin{cases} \frac{2}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases}\end{aligned}$$

Hence the Fourier series for $f(x)$ is:

$$
\begin{aligned}
f(x) &\sim \frac{1}{2} + \sum_{odd\ n} \frac{2}{n\pi} \sin(nx) = \\
&= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} \sin(2n+1)x, \quad \forall x \in (-\pi, \pi).
\end{aligned}
$$

Example 42 *Let $f(x) = x$, for all $x \in [-\pi, \pi)$, and $f(x)$ be 2π-periodic. We will compute the Fourier coefficients for this function. Notice that $\cos(nx)$ is an even function, while f and $\sin(nx)$ are odd functions.*

Solution:

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi} \int_{-\pi}^{\pi} xdx = 0 \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos(nx)dx = 0 \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x\sin(nx)dx = \\
&= \frac{2}{\pi} \int_{0}^{\pi} x\sin(nx)dx = \frac{2}{\pi} \left(\left[-\frac{x\cos(nx)}{n} \right]_0^{\pi} + \left[\frac{\sin(nx)}{n^2} \right]_0^{\pi} \right) = (-1)^{n+1}\frac{2}{n}
\end{aligned}
$$

Notice that a_0, a_n are 0 because x and $x\cos(nx)$ are odd functions. Hence the Fourier series for $f(x) = x$ is:

$$
\begin{aligned}
x &= a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx), \quad \forall x \in (-\pi, \pi)
\end{aligned}
$$

Example 43 *Let $f(x) = x^2$, $x \in [-\pi, \pi)$, and $f(x)$ be 2π-periodic.*

Solution: Since f is *even* ($f(x) = f(-x)$ for all x), then $b_n = 0$.

$$
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{2}{3}\pi^2,
$$

and for $n \geq 1$

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx \\
&= \frac{1}{n\pi} \left\{ x^2 \sin(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\sin(nx)\, dx \right\} \\
&= \frac{(-1)^n \cdot 4}{n^2}.
\end{aligned}
$$

Thus for $x \in (-\pi, \pi)$,

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{n^2} \cos(nx).$$

Points of discontinuity and convergence

In equation (14), "=" means that the series on the right converges to the function on the left at each point x. It often happens that the Fourier series of a function f fails to converge to that function, in particular at the points of discontinuity of f.

The facts are:

- **The Fourier Theorem**: If the function $f(x)$ is piecewise continuously differentiable then its Fourier series converges for every x to the average value

$$f_{av}(x) = \frac{f(x+) + f(x-)}{2}, \tag{13}$$

  where

$$f(x+) = \lim_{t \to x+} f(t), \quad f(x-) = \lim_{t \to x-} f(t).$$

- At the points where $f(x)$ is continuous, $f_{av}(x) = f(x)$.

All the functions we shall consider in the sequel are piecewise continuously differentiable, and therefore the Fourier series will represent the function. In order to ensure that the Fourier series of function $f(x)$ converges to that function at every $x \in \mathbb{R}$, sometimes it is necessary to redefine $f(x)$ at the points of discontinuity x, so that $f_{av}(x) = f(x)$.

In the example above we notice that at the points of discontinuity $\pm n\pi$, $n = 0, 1, 2, ...$, the average value $f_{av}(\pm n\pi) = 0.5$, whereas the value of the function is 1 for n even and 0 for n odd. Thus, we need to redefine the values of f to be 0.5 at these points.

8.2 Fourier series of functions on finite intervals

Case 1: Full extension

Let $f(x)$ be defined on (a, b). Then we can extend $f(x)$ to a periodic function $\tilde{f}(x)$ with period $b - a$. Let $2L = b - a$, then $L = \frac{b-a}{2}$.

Definition 7 *The Fourier Series of $f(x)$ on (a,b) is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})) \tag{14}$$

at every $x \in [a,b]$ where f is continuous. The coefficients a_n $(n \geq 0)$ and the coefficients b_n $(n \geq 1)$ are calculated as follows:

$$a_n = \frac{1}{L}\int_a^b f(x)\cos(\frac{n\pi x}{L})\,dx, \;\; n = 0,1,2,\ldots; \quad b_n = \frac{1}{L}\int_a^b f(x)\sin(\frac{n\pi x}{L})\,dx, \;\; n = 1,2,3,\ldots.$$

Example 44 *Let* $f(x) = \begin{cases} 3, & -1 < x < 0; \\ 2x, & 0 \leq x < 3; \end{cases}$ *Find its Fourier series.*

Solution: $L = 2$.

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{3} f(x)dx = 6, \\
a_n &= \frac{1}{2}\int_{-1}^{3} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{n\pi}\left(3\sin\frac{n\pi}{2} + 6\sin\frac{3n\pi}{2} + \frac{4}{n\pi}\cos\frac{3n\pi}{2} - \frac{4}{n\pi}\right), \\
b_n &= \frac{1}{2}\int_{-1}^{3} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{n\pi}\left(-3 + 3\cos\frac{n\pi}{2} - 6\cos\frac{3n\pi}{2} + \frac{4}{n\pi}\sin\frac{3n\pi}{2}\right).
\end{aligned}$$

Case 2: Half-range Expansions

Definition 8 *Given a function $f(x)$ defined on $(0,L)$.*

(i) We extend $f(x)$ as an odd function on $(-L,L)$, i.e.,

$$f_{odd}(x) = \begin{cases} f(x), & x \in (0,L); \\ -f(-x), & x \in (-L,0). \end{cases}$$

Then $a_n = 0$ for all n, (14) becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \tag{15}$$

which is called **Fourier sine series** *of f, where*

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx \qquad n = 1,2,3,\ldots.$$

(ii) We extend $f(x)$ as an even function on $(-L,L)$, i.e.,

$$f_{even}(x) = \begin{cases} f(x), & x \in (0,L); \\ f(-x), & x \in (-L,0). \end{cases}$$

Then $b_n = 0$ *for all* n*, (14) becomes*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) \tag{16}$$

which is called **Fourier cosine series** *of* f*, where*

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx \qquad n = 0, 1, 2, 3, \ldots.$$

The cosine and sine series here are known as HALF-RANGE EXPANSIONs.

Example 45 *Let* $f(x) = \begin{cases} x, & 0 \le x < 1. \\ 0, & 1 \le x < 2; \end{cases}$ *. Find (i) the Fourier sine series, (ii) Fourier cosine series, (iii) The Fourier series of* $f(x)$ *on* $(0, 2)$*.*

Solution: (i) Fourier sine series: for $m = 1, 2, 3, \ldots,$

$$\begin{aligned} b_m &= \frac{2}{L}\int_0^L f(x)\sin(\frac{m\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\sin(\frac{m\pi x}{2})\,dx \\ &= \int_0^1 x\sin(\frac{m\pi x}{2})\,dx = \left[-\frac{2}{m\pi}x\cos(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi x}{2})\right]_0^1 \\ &= -\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}). \end{aligned}$$

(ii) Fourier cosine series: for $m = 0, 1, 2, 3, \ldots,$

$$\begin{aligned} a_m &= \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\cos(\frac{m\pi x}{2})\,dx \\ &= \int_0^1 x\cos(\frac{m\pi x}{2})\,dx = \left[\frac{2}{m\pi}x\sin(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi x}{2})\right]_0^1 \\ &= \frac{2}{m\pi}\sin(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi}{2}) - \frac{4}{m^2\pi^2}. \end{aligned}$$

(iii) 2-periodic extension of f:

$$\tilde{f}(x) = \begin{cases} x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \\ \tilde{f}(x+2) = \tilde{f}(x), & \text{for any } x. \end{cases}$$

In this case, $L = 1$.

$$\begin{aligned} a_0 &= \int_0^1 x\,dx = \frac{1}{2}, \\ a_m &= \int_0^1 x\cos(m\pi x)\,dx = \frac{(-1)^m - 1}{(m\pi)^2}, \\ b_m &= \int_0^1 x\sin(m\pi x)\,dx = \frac{(-1)^m}{m\pi}. \end{aligned}$$

The Fourier series of $f(x)$ on $(0, 2)$ is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{(-1)^n}{n\pi}\sin(n\pi x)\right].$$