

# MATH1005H1(BIT2004A) - Test 4: 14:35-15:25, Nov 21, Friday

Total points: 15

Closed book, non-programmable calculators are allowed!

Name:

Student Number:

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1. (6 points) Multiple choices, no partial marks.

(i) Given the sequence  $a_n = \frac{9^n}{4^n}$ , then  $\lim_{n \rightarrow \infty} a_n =$

(a) 5    (b) 17/3    (c) 5/9    (d) 0    (e) Does not exist.

**Solution:** (e)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{9}{4}\right)^n = +\infty$ .

(ii) Given the sequence  $\{b_n\}$ :  $b_1 = 2$ ,  $b_{n+1} = \frac{1}{3}(b_n + 8)$ . Then  $\lim_{n \rightarrow \infty} b_n =$

(a)  $\frac{1}{3}$     (b)  $\frac{1}{5}$     (c) 3    (d) 5    (e) 4

**Solution:** (e). Let the limit be  $L$ . Then by the recursive relation,  $L = \frac{1}{3}(L+8) \Rightarrow L = 4$ .

(iii) Given the sequence  $c_n = \frac{e^n}{n}$ , then  $\lim_{n \rightarrow \infty} c_n =$

(a) 0    (b) 1    (c) 2    (d)  $e$     (e) Does not exist.

**Solution:** (e).

By L'Hospital's rule,  $\lim_{x \rightarrow \infty} \frac{e^x}{x} = \lim_{n \rightarrow \infty} \frac{e^n}{1} = +\infty \Rightarrow \lim_{n \rightarrow \infty} c_n$  does not exist.

(iv)  $\sum_{n=5}^{\infty} \frac{1}{n(n+1)} =$

(a)  $\frac{1}{2}$     (b)  $\frac{1}{3}$     (c)  $\frac{1}{4}$     (d)  $\frac{1}{5}$     (e)  $+\infty$ .

**Solution:** (d). Note that

$$\begin{aligned} \frac{1}{(n)(n+1)} &= \frac{1}{n} - \frac{1}{n+1}, \\ S_n &= \sum_{j=5}^n \frac{1}{(j)(j+1)} = \sum_{j=5}^n \left( \frac{1}{j} - \frac{1}{j+1} \right) \\ &= \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} + \frac{1}{7} - \frac{1}{8} + \dots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = \frac{1}{5} - \frac{1}{n+1}. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{5}.$$

2. (2 points) Find all the values of  $k$  such that the series  $\sum_{n=1}^{\infty} n^{2k-7}$  is convergent.

**Solution:** By  $p$ -series test,  $-(2k - 7) > 1$ ,  $2k < 6$ ,  $k < 3$ .

3. (3 points) Find the sum of the following series:  $\sum_{n=1}^{\infty} \frac{1 + 5^n}{9^n}$ .

**Solution:**

$$\sum_{n=1}^{\infty} \frac{1 + 4^n}{9^n} = \sum_{n=1}^{\infty} \frac{1}{9^n} + \sum_{n=1}^{\infty} \frac{5^n}{9^n} = \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^n = \frac{\frac{1}{9}}{1 - \frac{1}{9}} + \frac{\frac{5}{9}}{1 - \frac{5}{9}} = \frac{11}{8} = 1.375.$$

4. (4 points) Using **Integral Test**, determine if the following series converges or diverges:

$$\sum_{n=3}^{\infty} \frac{4}{n(\ln n)^5}.$$

**Solution:** We use Integral Test. Let  $f(x) = \frac{4}{x(\ln x)^5}$ ,  $x \geq 3$ . Then  $f(x) > 0$ , and continuous.

$$f'(x) = \frac{-4[(\ln x)^5 + 5(\ln x)^4]}{x^2(\ln x)^{10}} < 0,$$

$f(x)$  is decreasing. By substitution with  $u = \ln x$ , we have

$$\begin{aligned} \int_3^{\infty} \frac{4}{x(\ln x)^5} dx &= \lim_{b \rightarrow \infty} \int_3^b \frac{4}{x(\ln x)^5} dx \\ &= \lim_{b \rightarrow \infty} \int_{\ln 3}^{\ln b} \frac{4}{u^5} du \\ &= \left(\frac{1}{\ln 3}\right)^4. \end{aligned}$$

Hence it is convergent. By Integral Test, the series is convergent.