

MATH1005H1(BIT2004A) - Test 3: 14:35-15:25, Nov 7, Friday

Total points: 15

Closed book, non-programmable calculators are allowed!

Name:

Student Number:

- [3] 1. Let $y(x)$ be the solution of the differential equation: $y'y'' = 4x$, $y(0) = 1$, $y'(0) = 0$. Which of the following is a value of $y(3)$?

(a) 8 (b) -4 (c) 3 (d) 5 (e) 10

Solution: (e). Let $z = y'$. Then $zz' = 4x \Rightarrow \frac{1}{2}z^2 = 2x^2 + c$. Note that $z(0) = y'(0) = 0$, we have $c = 0$. Thus $z^2 = 4x^2 \Rightarrow z = \pm 2x \Rightarrow y' = \pm 2x, \Rightarrow y = \pm x^2 + d$. By $y(0) = 1$, $d = 1$. Thus $y = \pm x^2 + 1, \Rightarrow y(3) = 10$, or -8 .

- [3] 2. Solve the differential equation: $y''' - 3y'' - 4y' + 12y = 0$.

(a) $c_1e^{2x} + c_2e^{-2x} + c_3e^{-3x}$ (b) $c_1e^{-2x} + c_2xe^{-2x} + c_3e^{-3x}$ (c) $c_1e^{2x} + c_2xe^{2x} + c_3e^{3x}$
(d) $c_1e^{2x} + c_2e^{3x} + c_3e^{-3x}$ (e) $c_1e^{2x} + c_2e^{-2x} + c_3e^{3x}$

Solution: (e).

The indicial equation is $r^3 - 3r^2 - 4r + 12 = 0$, which gives $r = 2, -2, 3$.

Thus $y = c_1e^{2x} + c_2e^{-2x} + c_3e^{3x}$.

- [3] 3. Consider the differential equation $y'' - 3y' + 2y = e^{3x}$. Let $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ be a particular solution of the equation, where $y_1(x) = e^x, y_2(x) = e^{2x}$. Find $u_1(x)$.

(a) $e^{2x} + e^x$ (b) $\frac{1}{2}e^{2x}$ (c) $3e^x$ (d) e^x (e) $-\frac{1}{2}e^{2x}$

Solution: (e).

Note that the indicial equation of the homogeneous equation $y'' - 3y' + 2y$ is

$$r^2 - 3r + 2 = 0, \Rightarrow r = 1, 2, \Rightarrow y_1(x) = e^x, y_2(x) = e^{2x}.$$

From $y_1(x) = e^x$, $y_2(x) = e^{2x}$, $f(x) = e^{3x}$ we have

$$u_1'(x) = \frac{-y_2 f}{y_1 y_2' - y_1' y_2} = \frac{-e^{2x} e^{3x}}{2e^x e^{2x} - e^x e^{2x}} = -e^{2x}, \Rightarrow u_1(x) = -\frac{1}{2}e^{2x}.$$

$$u_2'(x) = \frac{y_1 f}{y_1 y_2' - y_1' y_2} = \frac{e^x e^{3x}}{2e^x e^{2x} - e^x e^{2x}} = e^x, \Rightarrow u_2(x) = e^x.$$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = -\frac{1}{2}e^{2x}e^x + e^x e^{2x} = \frac{1}{2}e^{3x}.$$

- [6] 4. Find the general solution to the system: $\frac{dx}{dt} = 2x + 3y$, $\frac{dy}{dt} = x + 4y$.

Solution: In matrix form, the system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 3 \\ 1 & 4 - \lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 5. \quad 2 \text{ points}$$

$$\text{For } \lambda_1 = 1, \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0} \Rightarrow a + 3b = 0, b = 1 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}. \quad 1 \text{ point}$$

$$\text{For } \lambda_2 = 5, \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0} \Rightarrow a - b = 0, b = 1 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad 1 \text{ point}$$

Thus, $\mathbf{x}_1(t) = e^t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2(t) = e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are independent solutions, and the general solution is $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$x(t) = -3c_1 e^t + c_2 e^{5t}, \quad y(t) = c_1 e^t + c_2 e^{5t}. \quad 2 \text{ points}$$