

MATH1005H1(BIT2004A) - Test 1 - 2:35 - 3:25pm, Oct 3, Friday

Total points: 15

Closed book, non-programmable calculators are allowed!

Name:

Student Number:

- [3] 1. Solve the initial value problem: $\frac{dy}{dx} = \frac{xe^x}{3y^2}, y(0) = 2$.

Solution: By separating x and y we have

$$3y^2 dy = xe^x dx \Rightarrow \int 3y^2 dy = \int xe^x dx \Rightarrow y^3 = xe^x - e^x + C.$$

$$y(0) = 2 \Rightarrow C = 9; \Rightarrow y^3 = xe^x - e^x + 9.$$

- [3] 2. Solve the following differential equation $x \frac{dy}{dx} + 2y = \frac{\cos x}{x}$.

Solution: Write the equation as

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}.$$

$$I(x) = e^{\int 2/x dx} = e^{\ln x^2} = x^2.$$

Multiply the two sides by x^2 :

$$x^2 \frac{dy}{dx} + 2xy = \cos x. \Rightarrow$$

$$(x^2 y)' = \cos x, \Rightarrow x^2 y = \sin x + C, \Rightarrow y = x^{-2} \sin x + Cx^{-2}.$$

- [3] 3. Find an integrating factor which makes the equation $12xy^2 + 3y - 6 + (6x^2y + x) \frac{dy}{dx} = 0$ exact.

Solution: Let

$$P(x, y) = 12xy^2 + 3y - 6, Q(x, y) = 6x^2y + x. \Rightarrow \frac{P_y - Q_x}{Q} = \frac{2}{x}.$$

$$\frac{I'(x)}{I(x)} = \frac{P_y - Q_x}{Q} = \frac{2}{x}. \Rightarrow I(x) = x^2.$$

- [3] 4. Change the equation $\frac{dy}{dx} + e^{2x}y = y^7x^2$ to a linear equation (do not solve the equation!).

Solution: This is a Bernoulli DE with $n = 7$. Let $u = y^{1-n} = y^{-6}$. (1 point)

Then $u' = -6y^{-7}y', \Rightarrow y' = -\frac{1}{6}y^7u'$.

Substitute this into the DE we have

$$-\frac{1}{6}y^7u' + e^{2x}y = y^7x^2.$$

Divide two sides by y^7 we have

$$-\frac{1}{6}u' + e^{2x}u = x^2, \Rightarrow u' - 6e^{2x}u = -6x^2. \quad (2\text{points})$$

- [3] 5. Solve the following exact differential equation: $2xy^3 + (3x^2y^2 + 4)\frac{dy}{dx} = 0$.

Solution: Let

$$P(x, y) = 2xy^3, Q(x, y) = 3x^2y^2 + 4.$$

Since the equation is exact, there exists a potential function $f(x, y)$ such that

$$P(x, y) = f_x(x, y), Q(x, y) = f_y(x, y).$$

From $P(x, y) = f_x$ we have $f_x = 2xy^3$. Integrate two sides according to x we have

$$f(x, y) = x^2y^3 + g(y).$$

The above result implies that

$$f_y(x, y) = 3x^2y^2 + g'(y).$$

Combining this with $Q(x, y) = f_y(x, y)$ we obtain

$$3x^2y^2 + g'(y) = Q(x, y) = 3x^2y^2 + 4, \Rightarrow$$

$$g'(y) = 4, \Rightarrow g(y) = 4y + C.$$

Thus $f(x, y) = x^2y^3 + 4y + C$. The solution is

$$x^2y^3 + 4y = C.$$