Systems of Linear Differential Equations

$y'(t) = a\,y(t)$

$\frac{dy}{dt} = ay$

$\int \frac{1}{y}\,dy = \int a\,dt$

$\ln y = at + C_1$

$y = e^{at + C_1}$

$= Ce^{at}$

C is a constant: $C = e^{C_1} = y(0)$

Consider two functions, $y_1(t)$ & $y_2(t)$, and system

$$(*)\ \begin{cases} y_1'(t) = 2y_1(t) + 5y_2(t) \\ y_2'(t) = y_1(t) - 2y_2(t) \end{cases}$$

In matrix form,

$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ $\quad \vec{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$

$(*)$ becomes $\underbrace{\vec{y}'(t)}_{\text{vector of functions}} = \underbrace{A}_{\text{matrix of constants}} \underbrace{\vec{y}}_{\text{vector of functions}}$

① Given any $n \times n$ constant matrix A, if $\vec{v}$ is an eigenvec with eigenval: λ, then

$\vec{y} = e^{\lambda t}\vec{v}$ is a solution to $(*)$, $\vec{y}' = A\vec{y}$

② If $\vec{y_\alpha}$, $\vec{y_\beta}$ are solutions to $(*)$ and C_1, C_2 are constants, then

$C_1\vec{y_\alpha} + C_2\vec{y_\beta}$ is also a solution to $(*)$

$\boxed{\text{e.g.}}$ For $\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$, calcs give eigvec $\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ with eigen val $\lambda = 3$

$\vec{y} = e^{\lambda t}\vec{v} = e^{3t}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5e^{3t} \\ e^{3t} \end{bmatrix}$ is a solution to $(*)$

$\vec{y}' = A\vec{y} = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 5e^{3t} \\ e^{3t} \end{bmatrix} = 3\begin{bmatrix} 5e^{3t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 15e^{3t} \\ 3e^{3t} \end{bmatrix}$

$\Rightarrow \vec{y} = e^{\lambda t}\vec{v}$ is a solution to $\vec{y}'(t) = A\vec{y}$

$\boxed{\text{E.g}}$ $$(**)\ \begin{cases} y_1'(t) = 2y_1(t) + y_2(t) \\ y_2'(t) = y_1(t) + 2y_2(t) \end{cases} \qquad \vec{y}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{y}$$

ⓐ Find the general solution

ⓑ Solve $(**)$ if $y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

ⓐ $p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-3)(\lambda-1)$. eigvals 3, 1 w/ associated eigvecs $C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively

$e^{\lambda_1 t}V_1$, $e^{\lambda_2 t}V_2$ are solutions to $(**)$

The general solution is $\vec{y} = C_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{1t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

ⓑ $y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = C_1 e^{3(0)}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{(0)}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \end{array}\right]$ $\quad C_1 = 3/2,\ C_2 = -1/2$

$y(t) = \frac{3}{2}e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a particular solution to $(**)$ when $y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

* If A has real entries, then the real part and imaginary part of a solution to $\vec{y}' = A\vec{y}$ are also solutions.

From 2x2 e.g., $e^{\lambda_1 t}\vec{v}_1$ is a solution.

$$e^{\lambda_1 t} v_1 = e^{(2+i)t}\begin{bmatrix}1-i\\1\end{bmatrix}$$

$$= e^{2t}(\cos t + i\sin t)\begin{bmatrix}1-i\\1\end{bmatrix}$$

$$= \underbrace{e^{2t}\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix}}_{\text{real part}} + \underbrace{i e^{2t}\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}}_{\text{imaginary part}}$$

★ From *, $\vec{y}(t) = e^{2t}\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix}$ and $\vec{y}(t) = e^{2t}\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}$

are both solutions to $\vec{y}' = A\vec{y}$

Hence the general solution in real form is

$$y(t) = C_1 e^{2t}\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix} + C_2 e^{2t}\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}$$

From this, to obtain the solution for the initial value problem with $y(0)=\begin{bmatrix}3\\1\end{bmatrix}$ plug $t=0$,

$$\begin{bmatrix}3\\1\end{bmatrix} = C_1\begin{bmatrix}1\\1\end{bmatrix} + C_2\begin{bmatrix}-1\\0\end{bmatrix} \qquad \begin{matrix}e^0=1\\ \cos 0 = 1\\ \sin 0 = 0\end{matrix}$$

$$\left[\begin{array}{cc|c}1&-1&3\\1&0&1\end{array}\right] \to \left[\begin{array}{cc|c}1&0&1\\0&1&-2\end{array}\right] \qquad \begin{matrix}C_1=1\\C_2=-2\end{matrix}$$

Hence

$$y(t) = e^{2t}\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix} - 2e^{2t}\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}$$

$$= e^{2t}\begin{bmatrix}3\cos t-\sin t\\ \cos t-2\sin t\end{bmatrix}$$

Note: For $e^{\lambda_1 t}v_1$ and $e^{\lambda_2 t}v_2$, they have the same real parts and their imaginary parts are negative of each other.

* Summary: If A is 2x2 with eigs $\lambda_1, \lambda_2, v_1, v_2$, with system $\vec{y}' = A\vec{y}$

General Sol'n | Sol'n Satisfying given IV $y(0)=\begin{bmatrix}a\\b\end{bmatrix}$

Complex Form $C_1 e^{\lambda_1 t}v_1 + C_2 e^{\lambda_2 t}v_2$ $\xrightarrow{\text{Complex Linear System}}$ Complex Form

$\downarrow$ take real & imaginary parts of $e^{\lambda_1 t}, v_1$ | $\downarrow$ simplification

Real Form $\xrightarrow{\text{Real linear system}}$ Real Form

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e.g. $A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}$ $\vec{y}\,' = A\vec{y}$

System of Linear Diff Eq'ns, 3x3; 3 distinct, real eigenvalues

e.g. $\vec{y}\,' = A\vec{y}$ $\vec{y}\,' = \begin{cases} 3y_1 - 6y_2 - 7y_3 \\ y_1 + 8y_2 + 5y_3 \\ -1y_1 - 2y_2 + y_3 \end{cases} \Rightarrow \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix} \vec{y}$

(i) Find the general solution

(ii) Find the solution if $y(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

i) By calcs, $p(\lambda) = -(\lambda-2)(\lambda-4)(\lambda-6)$

eval	corresponding evec
$\lambda_1 = 2$	$[1\ -1\ 1]^T$
$\lambda_2 = 4$	$[-1\ -1\ 1]^T$
$\lambda_3 = 6$	$[2\ 1\ 0]^T$

General Solution:

$$\vec{y} = e^{2t} c_1 [1\ -1\ 1]^T + e^{4t}[-1\ -1\ 1]^T + e^{6t}[2\ 1\ 0]^T$$

ii) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{array}{cccc} & & & \\ \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{array}\right] \\ v_1 \quad v_2 \quad v_3 \quad y(0) \end{array} \rightarrow \begin{array}{l} c_1 = -1 \\ c_2 = c_3 = 1 \end{array}$$

$$\vec{y} = -e^{2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + e^{4t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + e^{6t}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

System of Linear Diff Eq'ns; 2x2; 2 distinct non-real eigenvalues

eg $\vec{y}' = A\vec{y}$ $\quad \vec{y}' = \begin{cases} y_1 + 2y_2 \\ -y_1 + 3y_2 \end{cases} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

i) Find the general solution

ii) Find the solution if $y(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

i) $p(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 5$

$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\quad \lambda = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm \frac{2i}{2} = 2 \pm i$

$\lambda_1 = 2 + i$; $\lambda_2 = 2 - i$

For $\lambda_1 = 2+i$:

$\left[\begin{array}{cc|c} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 1-i & 0 \\ -1-i & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ -1-i & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$

* $② \rightarrow ② - (-1-i)①$

$2 - (-1-i)(-1+i)$

$= 2 - [1+1] = 2 - 2 = 0$

Let $x_2 = t_1$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (1-i)t \\ t \end{bmatrix} = t \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ is an eigenvec with eigenval $\lambda_1 = 2+i$

Note: A has (all) real entries; take complex conjugate

$\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is an eigenvec with eigenval $\lambda_2 = 2-i$

General Solution in complex form: $\quad \vec{y} = c_1 e^{(2+i)t} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} + c_2 e^{(2-i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

ii) General solution was $\vec{y} = C_1 e^{(2+i)t} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} + C_2 e^{(2-i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

To get solution, we need $y(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Putting t into the general sol'n,

$$c_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1-i & 1+i & 3 \\ 1 & 1 & 1 \end{array}\right]$$

$$① \to ① \times (1+i) \left[\begin{array}{cc|c} 2 & 2i & 3+3i \\ 1 & 1 & 1 \end{array}\right]$$

$$② \to ② - \tfrac{1}{2}① \left[\begin{array}{cc|c} 2 & 2i & 3+3i \\ 0 & 1-i & -\frac{1}{2} - \frac{3}{2}i \end{array}\right]$$

$$(1-i)C_2 = -\frac{1}{2} - \frac{3}{2}i$$

$$C_2 = \frac{-\frac{1}{2} - \frac{3}{2}i}{1-i} \left(\frac{1+i}{1+i}\right)$$

$$C_2 = \frac{1-2i}{2} = \frac{1}{2} - i$$

$$2C_1 + 2iC_2 = 3 + 3i$$

$$2C_1 + 2i\left(\tfrac{1}{2} - i\right) = 3 + 3i$$

$$C_1 = \frac{1}{2} + i$$

the solution in complex form is

$$\vec{y(t)} = \left(\tfrac{1}{2} + i\right) e^{(2+i)t} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} + \left(\tfrac{1}{2} - i\right) e^{(2-i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Using the fact that $e^{(2+i)t} = e^{2t+it}$

$$= e^{2t}(\cos t + i \sin t)$$

one can simplify the complex form above to real form (the imaginary parts cancel each other) and get

$$\vec{y(t)} = e^{2t} \begin{bmatrix} 3\cos t - \sin t \\ \cos t - 2\sin t \end{bmatrix}$$

12.3) $\frac{d}{dx}\vec{x}(t) = \begin{bmatrix} -7 & -6 \\ 9 & 8 \end{bmatrix} \vec{x}(t)$ Find general sol'n.

$\begin{vmatrix} -7-\lambda & -6 \\ 9 & 8-\lambda \end{vmatrix} = \lambda^2 - \lambda - 56 + 54$ $(\lambda-2)(\lambda+1)$

$\lambda^2 - \lambda - 2$ $\lambda = 2, -1$

$\lambda=2$ $\begin{bmatrix} -9 & -6 \\ 9 & 6 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 1 & 2/3 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2/3\, s \\ s \end{bmatrix}$

$\lambda=-1$ $\begin{bmatrix} -6 & -6 \\ 9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix}$

$\vec{y}(t) = c_1 e^{2t} \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

12.4) General sol'n in <u>Real</u> Form $x'(t) = 2x(t) - y(t)$

$y'(t) = 5x(t) - 2y(t)$

$\begin{vmatrix} 2-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 5$

$\lambda^2 + 1$ $\sqrt{-1} = \sqrt{i^2}$

$\lambda^2 = -1$ $\lambda = i, -i$ $\pm i$

$\lambda_1 = i$ $\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix}$ ② → ② − (−2−i) ① → $\begin{bmatrix} & \\ 0 & \end{bmatrix}$ No

Get rid of left-most column

$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix}$ ① → ① − $\frac{1}{5}$ ②(2−i) → $\begin{bmatrix} 0 & 0 \\ 5 & -2-i \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s(2+i) \\ s \end{bmatrix} = s \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

$\frac{1}{5}(-2-i)(2-i)$

$\frac{1}{5}(i^2 - 4) = -1$

$\vec{y}(t) = c_1 e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

$= c_1 (\cos t + i \sin t) \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 (\cos(-t) + i \sin(-t)) \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

12.4) General solution IN REAL FORM

$$x'(t) = 2x(t) - y(t)$$
$$y'(t) = 5x(t) - 2y(t)$$

$$\begin{vmatrix} 2-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 5 = \lambda^2 + 1 \qquad \lambda^2 = -1 \qquad \lambda = \pm i \qquad \sqrt{-1} = i^2 = \pm i$$

$$\lambda_1 = i \quad \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \quad ② \to ② - (2-i)① \to \begin{bmatrix} & \\ & 0 \end{bmatrix}$$ No. Get rid of left-most col

$$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} ① \to ① - \tfrac{1}{5}②(2-i) \to \begin{bmatrix} 0 & 0 \\ 5 & -2-i \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} \frac{2+i}{5} \\ 1 \end{bmatrix}$$

$$\tfrac{1}{5}(-2-i)(2-i)$$
$$\tfrac{1}{5}(i^2-4) = -1$$

$$\vec{y(t)} = c_1 e^{it}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 e^{-it}\begin{bmatrix} 2-i \\ 1 \end{bmatrix} \qquad e^{(a+ib)t} = e^{at}(\cos(bt) + i\sin(bt))$$
$$= c_1\begin{bmatrix} \frac{2}{5}+\frac{1}{5}i \\ 1 \end{bmatrix}(\cos(t) + i\sin(t)) + c_2\begin{bmatrix} \frac{2}{5}-\frac{1}{5}i \\ 1 \end{bmatrix}(\cos(-t) + i\sin(-t))$$

$$c_1\left(e^{a_1 t}\cos(b_1 t)\,\mathrm{Re}(\vec{x_1}) - e^{a_1 t}\sin(b_1 t)\,\mathrm{Im}(\vec{x_1})\right)$$
$$+ c_2\left(e^{a_2 t}\sin(b_2 t)\,\mathrm{Re}(\vec{x_2}) + e^{a_2 t}\cos(b_2 t)\,\mathrm{Im}(\vec{x_2})\right)$$

$$c_1\left(\begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}\cos(t) - \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix}\sin(t)\right)$$
$$+ c_2\left(\begin{bmatrix} 2/5 \\ 1 \end{bmatrix}\sin(-t) + \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}\cos(-t)\right)$$

Another Example

$$\begin{cases} x'(t) = 2x(t) - 3y(t) \\ y'(t) = 6x(t) - 4y(t) \end{cases} \quad \cdots \quad (*)$$

(a) Find general solution in <u>real</u> form

(b) Find solution if $x(0)=0$, $y(0)=2$ (in real form)

(a) $\vec{x}' = A\vec{x}$ $\quad A = \begin{bmatrix} 2 & -3 \\ 6 & -4 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$\det \begin{vmatrix} 2-\lambda & -3 \\ 6 & -4-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 8 + 18 = \lambda^2 + 2\lambda + 10$$

$$\lambda = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm \frac{\sqrt{-36}}{2} = -1 \pm \frac{6i}{2} = -1 \pm 3i$$

$\lambda_1 = -1+3i$

$$\left[\begin{array}{cc|c} 3-3i & -3 & 0 \\ 6 & -3-3i & 0 \end{array}\right]$$

$$\vec{y} = e^{\lambda t} \vec{v}$$

12.1) $\vec{y}' = A\vec{y}$

$x'(t) = x(t) + 6y(t)$
$y'(t) = -x(t) + 6y(t)$

$$A = \begin{bmatrix} 1 & 6 \\ -1 & 6 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 6 \\ -1 & 6-\lambda \end{vmatrix} = \begin{matrix} \lambda^2 - 7\lambda + 6 + 6 \\ \lambda^2 - 7\lambda + 12 \end{matrix} = (\lambda - 4)(\lambda - 3)$$

$\lambda_1 = 3 \qquad \lambda_2 = 4$

$\lambda_1 = 3$ $\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3s \\ s \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is eigenvector of eigenvalue 3

$\lambda_2 = 4$ $\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2s \\ s \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ eigvec of eigval 4

$$\vec{y} = c_1 e^{3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\vec{y}' = A\vec{y}$ $\qquad$ $\vec{y} = e^{\lambda t}\vec{v}$

12.2) $x'(t) = 3x(t) - 5y(t)$
$y'(t) = 4x(t) - 6y(t)$

$x(0) = 7 \quad y(0) = 6$

$$\begin{vmatrix} 3-\lambda & -5 \\ 4 & -6-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 18 + 20$$
$$\lambda^2 + 3\lambda + 2 \qquad (\lambda+2)(\lambda+1)$$

$\lambda_1 = -2 \qquad \lambda_2 = -1$

$\lambda_1 = -2$ $\begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix} \to \begin{bmatrix} 5 & -5 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix}$

$1 \ -5/4$

$\lambda_2 = -1$ $\begin{bmatrix} 4 & -5 \\ 4 & -5 \end{bmatrix} \to \begin{bmatrix} 4 & -5 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5/4\, s \\ s \end{bmatrix}$

$$\vec{y}(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-1t}\begin{bmatrix} 5/4 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 5/4 \\ 1 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 5/4 & 7 \\ 1 & 1 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1/4 & 1 \\ 1 & 1 & 6 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \end{array}\right]$

$c_1 = 2 \qquad c_2 = 4$

$$\vec{y}(t) = 2e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4e^{-t}\begin{bmatrix} 5/4 \\ 1 \end{bmatrix}$$

$x(t) = 2e^{-2t} + 5e^{-t}$
$y(t) = 2e^{-2t} + 4e^{-t}$