Matrix Powers

Let $D = \begin{bmatrix} 3 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ Through calcs, $p(\lambda) = -(\lambda-1)^2(\lambda-2)$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are lin. indept. evecs w/ evals 1

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvec w/ eigenval 2.

Rmk $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an eigenbasis of D.

Find $D^{10} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $D^{10} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

i) $D \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow D^k \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 2^k \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow D^{10} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 2^{10} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2048 \\ 1024 \\ 1024 \end{bmatrix}$

ii) $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is not an eigenvector but it is a lin comb. of eigenvectors.

$$\begin{aligned} D^{10} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} &= D^{10} \left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= D^{10} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 2 D^{10} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= (2)^{10} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 2(1)^{10} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2048 \\ 1024 \\ 1024 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2048 \\ 1026 \\ 1024 \end{bmatrix} \end{aligned}$$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad A^8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ?$ $\quad A^8 = ?$

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evals $\quad p(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm \frac{\sqrt{4}\sqrt{-1}}{2} = 1 \pm i$$

The eigenvalues are $\lambda = 1-i$ and $1+i$

evecs $\quad$ For $\lambda = 1-i$, $A - I(1-i) = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \xrightarrow{R_1 \to R_1 - iR_2} \begin{bmatrix} 0 & -1-i^2 \\ 1 & i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} -i \\ 1 \end{bmatrix}$

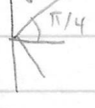
For $\lambda = 1+i$, $A - I(1+i) = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} 0 & -1-i^2 \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 2i \\ 0 \end{bmatrix}\left(\frac{1}{2i}\right)$$

$$A^8\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A^8\left[\left(\frac{1}{2i}\right)\left(\begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)\right]$$

$$= \left(\frac{1}{2i}\right)\left[A^8 \begin{bmatrix} i \\ 1 \end{bmatrix} - A^8 \begin{bmatrix} -i \\ 1 \end{bmatrix}\right]$$

$$= \left(\frac{1}{2i}\right)\left[(1+i)^8 \begin{bmatrix} i \\ 1 \end{bmatrix} - (1-i)^8 \begin{bmatrix} -i \\ 1 \end{bmatrix}\right]$$



arctan$(1) = \pi/4$, $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$$= \left(\frac{1}{2i}\right)\left[\left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^8 \begin{bmatrix} i \\ 1 \end{bmatrix} - \left(\sqrt{2}e^{-\frac{\pi i}{4}}\right)^8 \begin{bmatrix} -i \\ 1 \end{bmatrix}\right]$$

$$= \left(\frac{1}{2i}\right)\left(\sqrt{2}^8 e^{2\pi i} \begin{bmatrix} i \\ 1 \end{bmatrix} - \sqrt{2}^8 e^{-2\pi i} \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$$

$e^{2\pi i} = 1$

$$= \frac{1}{2i}\left(2^4 \begin{bmatrix} i \\ 1 \end{bmatrix} - 2^4 \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$$

$$= \frac{1}{2i}\begin{bmatrix} 32i \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix}$$

One similarly computes $A^8\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$

$$A^8 = \begin{bmatrix} | & | \\ A^8(e_1) & A^8(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

Hilroy

Matrix Powers - Applications to Random WALKS

Let P be an $n \times n$ transition matrix with all entries > 0.

① 1 is an eigenvalue of P and

(1.5) there is a unique probability vector $\vec{k}$ such that $P\vec{k} = \vec{k}$

② All other evalues of P, $|\lambda| < 1$

③ For any probability vector $\vec{x_0}$

$$\lim_{n\to\infty} P^n \vec{x^0} = \vec{k}, \quad \text{independent of } \vec{x_0}$$

$\vec{k}$ is called the **equilibrium probability.**

$\boxed{\text{E.g}}$ $P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/2 \end{bmatrix}$ Find x^n and $\lim_{n\to\infty} P^n x_0$ if i) $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ii) $x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$p(\lambda) = \det(P - \lambda I) = -\lambda(\lambda-1)(\lambda - 1/4)$$

$\lambda_1 = 1$ evec: $[4 \ 3 \ 5]^T$

$\lambda_2 = 1/4$ evec: $[1 \ 0 \ -1]^T$

$\lambda_3 = 0$ evec: $[0 \ -1 \ 1]^T$

i) $P^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P^n \left(c_1 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$ $\quad \left[\begin{array}{ccc|c} 4 & 1 & 0 & 1 \\ 3 & 0 & -1 & 0 \\ 5 & -1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/12 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 3/4 \end{array}\right]$

because linear

$$= P^n \left(\tfrac{1}{12} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} + \tfrac{2}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \tfrac{3}{4} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$= \tfrac{1}{12} P^n \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} + \tfrac{2}{3} P^n \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \tfrac{3}{4} P^n \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \tfrac{1}{12} (1)^n \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} + \tfrac{2}{3} \left(\tfrac{1}{4}\right)^n \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \tfrac{3}{4} 0 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lim_{n\to\infty} P^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \tfrac{1}{12} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/4 \\ 5/12 \end{bmatrix}$$

ii) $P^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P^n \left(\tfrac{1}{12} (1)^n \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} - \tfrac{1}{3} \left(\tfrac{1}{4}\right)^n \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0 \right)$ $\quad \left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 5 & -1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/12 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$

$$\lim_{n\to\infty} P^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/4 \\ 5/12 \end{bmatrix}$$

$\boxed{\text{Eg}}$ $P = \begin{bmatrix} 0.5 & 0.4 & 0.4 \\ 0.1 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$ $\quad x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ all entries in $P > 0$ so the theorem applies. Find $\lim_{n\to\infty} P^n \vec{x_0}$

By the theorem, 1 is an eigenvalue

$$P - I(\lambda = 1) = \left[\begin{array}{ccc|c} -0.5 & 0.4 & 0.4 & 0 \\ 0.1 & -0.8 & 0.2 & 0 \\ 0.4 & 0.4 & -0.6 & 0 \end{array}\right] \xrightarrow[\text{steps}]{} s \begin{bmatrix} 10/9 \\ 7/18 \\ 1 \end{bmatrix}$$ is an eigenvector of eigenvalue 1.

$$P\vec{k} = \lambda \vec{k}; \quad P\vec{k} = \vec{k}$$

$\vec{k} = s \begin{bmatrix} 10/9 \\ 7/18 \\ 1 \end{bmatrix}$ Find s s.t. k has sum of entries $== 1$

$$s(10/9 + 7/18 + 1) = 1, \quad s\left(\tfrac{5}{2}\right) = 1, \quad s = 2/5$$

$\vec{k} = \frac{2}{5} \begin{bmatrix} 10/9 \\ 7/18 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 7/45 \\ 2/5 \end{bmatrix}$

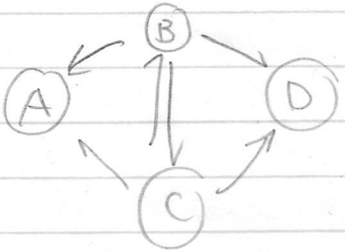
① $\vec{k}$ is an eigenvector of eigenvalue 1

② $\vec{k}$ is an equilibrium probability vector.

$\lim_{n\to\infty} P^n \vec{x_0} = \vec{k}$

Example where $\lim_{n\to\infty} P^n \vec{X_0}$ depends on $\vec{X_0}$

$$P = \begin{bmatrix} 1 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 1 \end{bmatrix}$$



Not all entries > 0, so the theorem does not apply.

evalue   evecs

$$\left.\begin{array}{ll} \lambda = 1 & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ \lambda = 0 & \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \\ \lambda = 1/2 & \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \end{array}\right\} \text{eigen basis}$$

$$\lim_{n\to\infty} P^n X_0 = \begin{cases} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \text{if } X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = V_1 \\ \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} & \text{if } X_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\left[V_1 + V_2 + V_3 - V_4\right] \\ & \phantom{\text{if }} X_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\left[V_1 + V_2 - V_3 - V_4\right] \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \text{if } X_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = V_2 \end{cases}$$

11.10) Consider a matrix A with eigvals $\lambda_1 = -0.6$, $\lambda_2 = -0.7$, $\lambda_3 = 1$ and

$\vec{v}_1 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ respective corresponding eigvecs.

Suppose $\vec{x} = -3\vec{v}_1 + 2\vec{v_2} - 2\vec{v_3}$.

a) Find an expression for $A^k \vec{x}$.

$$A^k\vec{x} = A^k\left(-3\ \vec{v}_1 + 2\vec{v}_2 - 2\vec{v}_3\right)$$

$$= -3A^k\vec{v}_1 + 2A^k\vec{v}_2 - 2A^k\vec{v}_3$$

$$= -3(-0.6)^k\vec{v}_1 + 2(-0.7)^k\vec{v_2} - 2(1)^k\vec{v}_3$$

$$= -3(-0.6)^k\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} + 2(-0.7)^k\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

b) Find $\lim_{k\to\infty} A^k\vec{x}$ $\quad -2\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -10 \\ -4 \end{bmatrix}$

11.11) Consider a simple model.

If you pass prev test, 0.3 chance you'll pass the next.

If you fail prev test, 0.6 chance you'll fail the next.

If it continues over a long time, what's the prob. you'll pass a test?

$$P = \begin{array}{cc} P & F \\ \left[\begin{array}{cc} 0.3 & 0.4 \\ 0.7 & 0.6 \end{array}\right] \end{array} \qquad \det(P - \lambda I) = (0.3-\lambda)(0.6-\lambda) - (0.4)(0.7)$$

$$= \lambda^2 - 0.9\lambda - 0.1 \quad = 0$$

$$10\lambda^2 - 9\lambda - 1 \quad = 0$$

$$(10\lambda + 1)(\lambda - 1) = 0 \qquad \lambda = 1, -0.1$$

$$\lambda = 1 \quad \left[\begin{array}{cc|c} -0.7 & 0.4 & 0 \\ 0.7 & -0.4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 7 & -4 & 0 \\ & & 0 \end{array}\right] \qquad \text{eigvec } \begin{bmatrix} 4/7\,s \\ s \end{bmatrix} = s\begin{bmatrix} 4/7 \\ 1 \end{bmatrix}$$

$$1 = s(4/7 + 1) \qquad 7/11\begin{bmatrix} 4/7 \\ 1 \end{bmatrix} = \begin{bmatrix} 28/77 \\ 7/11 \end{bmatrix}$$

$$s = 7/11$$

11.12) Consider the transition matrix $P = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.4 & 0 & 0.2 \\ 0.1 & 0.5 & 0.3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 5 & 5 & 5 \\ 4 & 0 & 2 \\ 1 & 5 & 3 \end{bmatrix}$

a) Find the eigvals and corresponding eigvecs of P

$$\det \begin{bmatrix} 0.5-\lambda & 0.5 & 0.5 \\ 0.4 & -\lambda & 0.2 \\ 0.1 & 0.5 & 0.3-\lambda \end{bmatrix} = -0.5 \begin{vmatrix} 0.4 & 0.2 \\ 0.1 & 0.3-\lambda \end{vmatrix} - \lambda \begin{vmatrix} 0.5-\lambda & 0.5 \\ 0.1 & 0.3-\lambda \end{vmatrix} - 0.5 \begin{vmatrix} 0.5-\lambda & 0.5 \\ 0.4 & 0.2 \end{vmatrix}$$

$$= -0.5\left[0.1\overset{0}{\cancel{2}} - 0.4\lambda - 0.\cancel{0}2 + \cancel{0.1} - 0.2\lambda - 0.\cancel{2}0\right] - \lambda\left[\lambda^2 - 0.8\lambda \overset{0.10}{\cancel{+0.15}} \underset{\cancel{-0.05}}{}\right]$$

$$= 0.30\lambda - \lambda^3 + 0.8\lambda^2 - 0.10\lambda$$

$$= -\lambda^3 + 0.8\lambda^2 + 0.2\lambda$$

$$= 10\lambda^3 - 8\lambda^2 - 2\lambda$$

$$= \lambda(10\lambda^2 - 8\lambda - 2)$$

$$= \lambda(10\lambda + 2)(\lambda - 1) \qquad \text{roots: } \lambda = 1, \tfrac{-2}{10}, 0$$

$$\lambda = 1 \quad \left[\begin{array}{ccc|c} -0.5 & 0.5 & 0.5 & 0 \\ 0.4 & -1 & 0.2 & 0 \\ 0.1 & 0.5 & -0.7 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = 0 \quad \left[\begin{array}{ccc|c} 0.5 & 0.5 & 0.5 & 0 \\ 0.4 & 0 & 0.2 & 0 \\ 0.1 & 0.5 & 0.3 & 0 \end{array}\right] \xrightarrow{\text{ref}} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\lambda = -0.2 \quad \left[\begin{array}{ccc|c} 0.7 & 0.5 & 0.5 & 0 \\ 0.4 & 0.2 & 0.2 & 0 \\ 0.1 & 0.5 & 0.5 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

b) If the initial state is $\vec{X}_0 = [0.3 \ \ 0.35 \ \ 0.35]^T$, find the n'th state $\vec{X}_n$.

$$\left[\begin{array}{ccc|c} 2 & -1/2 & 0 & 0.3 \\ 1 & -1/2 & -1 & 0.35 \\ 1 & 1 & 1 & 0.35 \end{array}\right] \qquad \begin{bmatrix} 0.3 \\ 0.35 \\ 0.35 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{4}{10}\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} - \frac{3}{10}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{X_n} = P^n\vec{X_0} = \frac{1}{4}(1)^n\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{5}(0)^n\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} - \frac{3}{10}(-0.2)^n\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{X_n} = 1/4\,[2 \ 1 \ 1]^T - 3/10\,(-0.2)^n\,[0 \ -1 \ 1]^T$$

c) Find $\lim_{n \to \infty} X_n$

$$\frac{1}{4}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/4 \end{bmatrix}$$