Eigenvalues & Eigenvectors

a number, λ, and a vector, $\vec{x}$ in $\mathbb{R}^n$ are said to be an **eigenvalue-eigenvector pair** if

① $\vec{x} \neq \vec{0}$ ② $A\vec{x} = \lambda\vec{x}$

If these are both true, λ is an eigenvalue and $\vec{x}$ is an eigen vector with eigenvalue λ

e.g. $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, 2 is an eigenvalue, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 2

e.g. $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, not a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector.

If $A\vec{x} = \lambda\vec{x}$, $A\vec{x} - \lambda\vec{x} = \vec{0}$

$(A-\lambda)\vec{x}$ does not make sense (subtracting # from a matrix)

$A\vec{x} - \lambda I_n\vec{x} = \vec{0}$

$(A - \lambda I_n)\vec{x} = \vec{0}$

Recall: For a square matrix, B $(n \times n)$, $\det B = 0 \iff B\vec{x} = 0$ has a non-zero sol'n $\vec{x}$

Upshot: If $\vec{x}$ is an eigenvector of A with eigenvalue λ, then $\vec{x}$ is a non-zero solution of $(A - \lambda I_n)\vec{x} = 0$

$\det(A - \lambda I_n) = 0$

Finding the eigenvalues and eigenvectors of an $n \times n$ matrix A:

① Solve λ for $\det(A - \lambda I_n) = 0$

The roots are eigenvalues of A

Rmk: $p(\lambda) = \det(A - \lambda I_n)$ is a degree n polynomial

= the characteristic polynomial.

: degree n, so A has at most n eigenvalues (roots λ)

② For each eigenvalue, the non-zero solutions for $(A - \lambda I_n)\vec{x} = \vec{0}$ are eigenvector with eigen value λ.

10.1) Determine if $\vec{v}$ is an eigenvector of the matrix A?

$x\vec{v} = A\vec{v}$ scalar multiple.

a) $A = \begin{bmatrix} -27 & 10 \\ -50 & 18 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ $\begin{bmatrix} -27 & 10 \\ -50 & 18 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -119 \\ -224 \end{bmatrix}$ No.

b) $A = \begin{bmatrix} -10 & 7 \\ -14 & 11 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -10 & 7 \\ -14 & 11 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ Yes.

c) $A = \begin{bmatrix} -11 & -2 \\ 6 & -4 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} -11 & -2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \end{bmatrix}$ Yes.

10.2) Determine if λ is an eigenvalue of the matrix A?

a) $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 1 & -3 & 2 \end{bmatrix}$ $\lambda = 5$ $(2-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 0 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)(1-\lambda)$ No.

b) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ -3 & 4 & -1 \end{bmatrix}$ $\lambda = -1$ $(-1-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & -5-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda)(-5-\lambda)$ Yes.

c) $A = \begin{bmatrix} 11 & -3 & -12 \\ 0 & -4 & 0 \\ 6 & -3 & -7 \end{bmatrix}$ $\lambda = -2$ $(11-\lambda)\begin{vmatrix} -4-\lambda & 0 \\ -3 & -7-\lambda \end{vmatrix} + 6\begin{vmatrix} -3 & -12 \\ -4-\lambda & 0 \end{vmatrix}$

$p(\lambda) = +(11-\lambda)(\lambda+4)(\lambda+7) + 6(12)(\lambda+4)$

$p(-2) = (13)(2)(5) + 72(2)$ NO.

10.3) Consider the matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ -5 & -4 & 0 \end{bmatrix}$

What is the eigenvalue associated with each eigenvector?

a) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ -5 & -4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -4 \end{bmatrix}$ $\lambda = -4$

b) $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ -5 & -4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\lambda = 0$

c) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ -5 & -4 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 5 \end{bmatrix}$ $\lambda = -5$

* the eigenvector $\vec{x} \neq \vec{0}$ EVER.
· but the resulting vector can $= \vec{0}$ because $\lambda = 0$

rref by Matlab

10.4) The matrix $\begin{bmatrix} -1 & 3 & -6 \\ -8 & 2 & 2 \\ -4 & 3 & -3 \end{bmatrix}$ has eigenvalues $-4, -1, 3$. Find the eigenvectors.

$$-4: \left[\begin{array}{ccc|c} 3 & 3 & -6 & 0 \\ -8 & 6 & 2 & 0 \\ -4 & 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$-1: \left[\begin{array}{ccc|c} 0 & 3 & -6 & 0 \\ -8 & 3 & 2 & 0 \\ -4 & 3 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 2s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$3: \left[\begin{array}{ccc|c} -4 & 3 & -6 & 0 \\ -8 & -1 & 2 & 0 \\ -4 & 3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2s \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

10.10) Find the eigenvalues and corresponding eigenvectors of $\begin{bmatrix} -7 & -4 \\ 12 & 7 \end{bmatrix}$

$(-7-\lambda)(7-\lambda)+48$

$= \lambda^2 - 1 = (\lambda+1)(\lambda-1)$. $\quad \lambda = 1, -1$

$$\left[\begin{array}{cc|c} -8 & -4 & 0 \\ 12 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -6 & -4 & 0 \\ 12 & 8 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}s \\ s \end{bmatrix} = s\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

10.11) Find the eigenvalues and correspond e.vectors of $\begin{bmatrix} -3 & 4 & -2 \\ 0 & -1 & 20 \\ 0 & 0 & 4 \end{bmatrix}$

$(4-\lambda)\begin{vmatrix} -3-\lambda & 4 \\ 0 & -1-\lambda \end{vmatrix} = +(4-\lambda)(\lambda+3)(\lambda+1)$

$\lambda = 4, -3, -1$

$$-1: \left[\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$-3: \left[\begin{array}{ccc|c} 0 & 4 & -2 & 0 \\ 0 & 2 & 20 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$4: \left[\begin{array}{ccc|c} -7 & 4 & -2 & 0 \\ 0 & -5 & 20 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2s \\ 4s \\ s \end{bmatrix} = s\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Find the eigen-values, eigen-vectors for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

evalues: $p(\lambda) = A - \lambda I_n = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

$$\det(A - \lambda I_n) = (2-\lambda)^2 - 1$$
$$= 4 - 4\lambda + \lambda^2 - 1$$
$$= \lambda^2 - 4\lambda + 3$$
$$= (\lambda - 3)(\lambda - 1)$$
$$\lambda = 3, 1$$

The roots of $p(\lambda) = 0$ are 3, 1

The eigenvalues of A are 1 and 3

evectors: For every eigenvalue, find the non-zero sol'n to $(A - \lambda I_n)\vec{x} = \vec{0}$

• For $\lambda = 1$, $(A - \lambda I_n) = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$(A - \lambda I_n)\vec{x} = 0 \iff \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

basic free

Let $x_2 = t$; $x_1 + x_2 = 0$, $x_1 + t = 0$, $x_1 = -t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of eigenvalue 1.

• For $\lambda = 3$, $(A - \lambda I_n) = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$(A - \lambda I_n)\vec{x} = 0 \iff \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

basic free

Let $x_2 = t$; $x_1 - x_2 = 0$, $x_1 = t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of eigenvalue 3

Find the eigenvalues, eigen vectors of $A = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}$

evalues:

$p(\lambda) = \det \begin{bmatrix} 3-\lambda & -6 & -7 \\ 1 & 8-\lambda & 5 \\ -1 & -2 & 1-\lambda \end{bmatrix} = \begin{matrix} (3-\lambda)[(8-\lambda)(1-\lambda)+10] \\ -(-6)[(1-\lambda)+5] \\ +7[-2+(8-\lambda)] \end{matrix}$

$= -\lambda^3 + 12\lambda^2 - 44\lambda + 48$

Solve for $p(\lambda) = 0 = -\lambda^3 + 12\lambda^2 - 44\lambda + 48$

guess & test. $p(2) = -8 + 48 - 88 + 48 = 0$

Factor Theorem If p is a polynomial with $p(a) = 0$, then $\lambda - a$ divides $p(\lambda)$

$\rightarrow \lambda - 2$ divides $p(\lambda)$

$$\begin{array}{r} -\lambda^2 + 10\lambda - 24 \\ \lambda - 2 \; \overline{) -\lambda^3 + 12\lambda^2 - 44\lambda + 48} \\ +\lambda^3 - 2\lambda^2 + 20\lambda - 48 \\ -10\lambda^2 + 24\lambda \end{array}$$

$p(\lambda) = (\lambda-2)(-\lambda^2 + 10\lambda - 24)$

$= -(\lambda-2)(\lambda^2 - 10\lambda + 24)$

$= -(\lambda-2)(\lambda-6)(\lambda-4)$

roots are 2, 4, 6

evectors:

For $\lambda = 2$ $\left[\begin{array}{ccc|c} 1 & -6 & -7 & 0 \\ 1 & 6 & 5 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

basic basic free

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector w/ eigenvalue 2

For $\lambda = 4$ $\left[\begin{array}{ccc|c} -1 & -6 & -7 & 0 \\ 1 & 4 & 5 & 0 \\ -1 & -2 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector w/ e.value 4.

For $\lambda = 6$ $\left[\begin{array}{ccc|c} -3 & -6 & -7 & 0 \\ 1 & 2 & 5 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

free

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is an e.vector w/ evalue 6.

10.12) Find the eigenvalues and corresponding eigenvectors of $\begin{bmatrix} 3 & 2 & -1 \\ -1 & 8 & -3 \\ -2 & 8 & -2 \end{bmatrix}$

Note that at least one will be a small integer.

$$(3-\lambda)\begin{vmatrix} 8-\lambda & -3 \\ 8 & -2-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & -3 \\ -2 & -2-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 8-\lambda \\ -2 & 8 \end{vmatrix}$$

$$(3-\lambda)\left[(8-\lambda)(-2-\lambda)+24\right] - 2\left[(\lambda+2)-6\right] - \left[-8+2(8-\lambda)\right]$$

$$(3-\lambda)\left[\lambda^2-6\lambda-16+24\right] - 2\lambda+8+8-16+2\lambda$$

$$(3-\lambda)(\lambda^2-6\lambda+8) \qquad p(2)=0$$

$$-\lambda^3+9\lambda^2-26\lambda+24$$

$$\begin{array}{r} -\lambda^2+7\lambda-12 \\ \lambda-2 \;\big)\; -\lambda^3+9\lambda^2-26\lambda+24 \\ +\lambda^3 \quad -2\lambda^2 \quad +14\lambda \quad -24 \\ -7\lambda^2+12\lambda \end{array}$$

$$p(\lambda) = -(\lambda-2)(\lambda^2-7\lambda+12)$$
$$= -(\lambda-2)(\lambda-4)(\lambda-3)$$

$$p(\lambda)=0 \text{ for } \lambda = 2, 3, 4$$

$$\lambda=2 \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & 6 & -3 & 0 \\ -2 & 8 & -4 & 0 \end{array}\right] \xrightarrow[\text{rref}]{\text{by Matlab}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 s \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\lambda=3 \quad \left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ -1 & 5 & -3 & 0 \\ -2 & 8 & -5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 s \\ 1/2 s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\lambda=4 \quad \left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ -1 & 4 & -3 & 0 \\ -2 & 8 & -6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Geometric Examples of Eigenvalues, Eigenvectors

Let $L: 3x = 4y$ in $\mathbb{R}^2$

Find the eigen values, eigen vectors of the following matrices.

$$A = \text{Proj}_L = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix} \qquad B = \text{Ref}_L = \begin{bmatrix} 7/25 & 24/25 \\ 24/25 & 7/25 \end{bmatrix}$$

$$C = \text{Rot}_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad D = \text{Rot}_{\pi/4} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

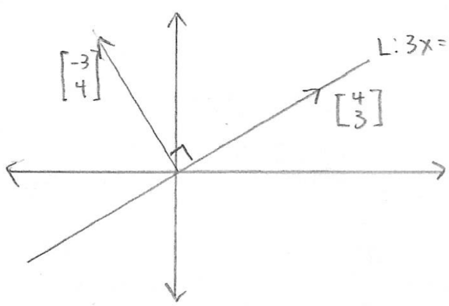
Proj

A: $p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 16/25 - \lambda & 12/25 \\ 12/25 & 9/25 - \lambda \end{vmatrix} = \lambda^2 - \lambda = \lambda(\lambda - 1)$, $\lambda = 0, 1$

$\lambda = 0$; $A - \lambda I_2 = A$

$\left[\begin{array}{cc|c} 16/25 & 12/25 & 0 \\ 12/25 & 9/25 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$; $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3/4\, t \\ t \end{bmatrix} = t \begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$; $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ is e.vec of eval 0

$\lambda = 1$; skip steps: $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 1.



$A \begin{bmatrix} -3 \\ 4 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} -3 \\ 4 \end{bmatrix} \perp L$

$A \begin{bmatrix} 4 \\ 3 \end{bmatrix} = (1) \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ because $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is on L

Ref

B: without $p(\lambda)$ calculation, let's think.. Reflection.

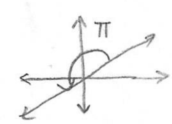
$B \begin{bmatrix} 4 \\ 3 \end{bmatrix} = (1) \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, because $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is on L, it is fixed by Ref_L

$B \begin{bmatrix} -3 \\ 4 \end{bmatrix} = (-1) \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ because $\begin{bmatrix} -3 \\ 4 \end{bmatrix} \perp L$, (-1)

degree 2 $p(\lambda)$
$\Rightarrow 1, -1$ are all the e-values of B.

Rot π

C: $p(\lambda) = \det(C - \lambda I_2) = (-1 - \lambda)^2 = (\lambda + 1)^2$, e-value is -1



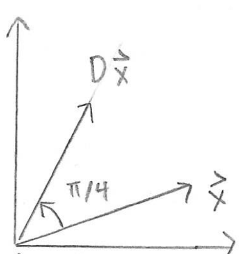
$C - \lambda I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; for $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For any $(s, t) \neq (0, 0)$, $\begin{bmatrix} s \\ t \end{bmatrix}$ is a non-zero solution and so is an evector of evalue -1

In particular, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are 2 l.i. evectors with evalue -1, agreeing with geometry.

Rot π/4

D:



$\vec{x}$, $D\vec{x}$ not in same, opp, or $\perp$ directions. No evals? Not true. This is a real picture.

$p(\lambda) = \begin{vmatrix} 1/\sqrt{2} - \lambda & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} - \lambda \end{vmatrix}$

$= \lambda^2 - \sqrt{2}\lambda + 1$

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{\sqrt{2} \pm \sqrt{-2}}{2}$

$= \dfrac{\sqrt{2} \pm i\sqrt{2}}{2}$

$\lambda = \sqrt{2}/2 \pm \sqrt{2}/2\, i$

$= 1/\sqrt{2} \pm 1/\sqrt{2}\, i$

Complex Conjugate Eigenvalue-Eigenvector

- For $D = Rot_{\pi/4} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\quad \lambda = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$

  two complex eigenvalues

- For $\lambda = 1/\sqrt{2} + 1/\sqrt{2}\, i$, $\quad A - \lambda I = \begin{bmatrix} -1/\sqrt{2}\, i & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2}\, i \end{bmatrix}$

$$\left[\begin{array}{cc|c} -1/\sqrt{2}\, i & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2}\, i & 0 \end{array}\right] \xrightarrow{\times\sqrt{2}} \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array}\right]$$

$R_2 \to R_2 + iR_1$

$$= \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$(-i) + i(1) = 0$

$(-1) + i(-i) = -1 + 1 = 0$

$-i^2 = 1$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\begin{bmatrix} i \\ 1 \end{bmatrix}$ is an evec of eval $1/\sqrt{2} + 1/\sqrt{2}\, i$

- For $\lambda = 1/\sqrt{2} - 1/\sqrt{2}\, i$
  i) Do the same procedure -OR-
  ii) Use the complex conjugate

$\overline{a+bi} = a - bi$

$\overline{zw} = \bar{z} \cdot \bar{w}$

$\overline{z+w} = \bar{z} + \bar{w}$

From above,

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Trick: take the complex conjugate

$$\begin{bmatrix} \overline{1/\sqrt{2}} & \overline{-1/\sqrt{2}} \\ \overline{1/\sqrt{2}} & \overline{1/\sqrt{2}} \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{1} \end{bmatrix} = \overline{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i} \begin{bmatrix} \bar{i} \\ \bar{1} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right) \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector with evalue $1/\sqrt{2} - 1/\sqrt{2}\, i$

* Rmk In general, if A is a square matrix with REAL entries, and if $\vec{x}$ is an complex evector of evalue λ, $\Rightarrow \overline{\vec{x}}$ is an evec of e-value $\bar{\lambda}$

10.5) $\begin{bmatrix} -5 & k \\ -5 & 2 \end{bmatrix}$ has 2 distinct real eigenvalues if and only if $k = ?$

$(-5-\lambda)(2-\lambda) - (-5)k$

$= (-10 + 3\lambda + \lambda^2) + 5k$

$= \lambda^2 + 3\lambda + (5k - 10)$

$b^2 - 4ac = 9 - 4(1)(5k-10)$

$= 9 - 20k + 40$

$= 49 - 20k$

$49 - 20k > 0$

$49/20 > k$

10.6) $\begin{bmatrix} 6 & k \\ -6 & 9 \end{bmatrix}$ has one repeated real eigenvalue for what value of k?

$(6-\lambda)(9-\lambda) - (-6)k$

$54 - 15\lambda + \lambda^2 + 6k$

$\lambda^2 - 15\lambda + 54 + 6k$

$b^2 - 4ac = 225 - 4(54+6k)$

$= 9 - 24k$

$9 - 24k = 0$

$k = 9/24$

10.9) Determine whether the eigenvalues of the matrices are distinct real, repeated real, or complex.

a) $\begin{bmatrix} -9 & -4 \\ 1 & -5 \end{bmatrix}$ $(-9-\lambda)(-5-\lambda) + 4$

$= 45 + 14\lambda + \lambda^2 + 4$

$= \lambda^2 + 14\lambda + 49$

$b^2 - 4ac = 14^2 - 4(49) = 0$ repeated real

b) $\begin{bmatrix} 2 & -3 \\ 3 & 8 \end{bmatrix}$ $(2-\lambda)(8-\lambda) + 9$

$= \lambda^2 - 10\lambda + 25$

$b^2 - 4ac = 10^2 - 4(25) = 0$ repeated real

c) $\begin{bmatrix} -4 & -5 \\ 5 & 3 \end{bmatrix}$ $(-4-\lambda)(3-\lambda) + 25$

$= \lambda^2 + \lambda + 13$

$b^2 - 4ac = 1^2 - 4(13) < 0$ complex

11.1) Give an example of a 2×2 real matrix without any real eigenvalue.

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = ad - \lambda(a+d) + \lambda^2 - bc$$
$$= \lambda^2 - \lambda(a+d) + (ad-bc)$$

$$\sqrt{b^2-4ac} = \sqrt{(a+d)^2 - 4(ad-bc)}$$
$$= \sqrt{a^2 + \cancel{2ad} + d^2 - \cancel{4}ad + 4bc}$$
$$= \sqrt{a^2+d^2-2ad+4bc}$$
$$a^2+d^2-2ad+4bc < 0$$

$$\begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix}$$

11.6) Find the e.vals & e.vecs of matrix $\begin{bmatrix} -2 & -10 \\ 4 & 10 \end{bmatrix}$

$$\begin{vmatrix} -2-\lambda & -10 \\ 4 & 10-\lambda \end{vmatrix} = (-2-\lambda)(10-\lambda) + 40$$
$$= -20 - 8\lambda + \lambda^2 + 40$$
$$0 = \lambda^2 - 8\lambda + 20$$
$$\lambda = \frac{8 \pm \sqrt{-16}}{2} = 4 \pm \frac{\sqrt{-1(4)^2}}{2} = 4 \pm \frac{4i}{2}$$
$$= 4 \pm 2i$$

$\lambda = 4+2i$
$$\begin{bmatrix} -6-2i & -10 \\ 4 & 6-2i \end{bmatrix} \to \begin{bmatrix} -24-8i & -40 \\ 24 & 36-12i \end{bmatrix} \to \begin{bmatrix} -8i & -4-12i \\ 8 & 12-4i \end{bmatrix} \to \begin{bmatrix} 0 & -4-4i^2 \\ 8 & 12-4i \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & \frac{3}{2}-\frac{1}{2}i \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s(-\frac{3}{2}+\frac{1}{2}i) \\ s \end{bmatrix} = s\begin{bmatrix} -\frac{3}{2}+\frac{1}{2}i \\ 1 \end{bmatrix}$$

Trick: Take the complex conjugate

$$\begin{bmatrix} -2 & 10 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} -\frac{3}{2}+\frac{1}{2}i \\ 1 \end{bmatrix} = (4+2i)\begin{bmatrix} -\frac{3}{2}+\frac{1}{2}i \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 10 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} -\frac{3}{2}-\frac{1}{2}i \\ 1 \end{bmatrix} = (4-2i)\begin{bmatrix} -\frac{3}{2}-\frac{1}{2}i \\ 1 \end{bmatrix}$$

112 Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 3 & -6 & 2 \\ 2 & -5 & 2 \\ 2 & -6 & 3 \end{bmatrix}$

$$\det \begin{bmatrix} 3-\lambda & -6 & 2 \\ 2 & -5-\lambda & 2 \\ 2 & -6 & 3-\lambda \end{bmatrix} = (3-\lambda)\begin{vmatrix} -5-\lambda & 2 \\ -6 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} -6 & 2 \\ -6 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & 2 \\ -5-\lambda & 2 \end{vmatrix}$$

$$= (3-\lambda)\left[(3-\lambda)(-5-\lambda)+12\right] - 2\left[6\lambda \overset{-6}{\cancel{-18+12}}\right] + 2\left[\overset{-2}{\cancel{-12+10}}+2\lambda\right]$$

$$= -(\lambda^2-6\lambda+9)(\lambda+5) + 36 - 12\lambda - 12\lambda + 12 - 4 + 4\lambda$$

$$= -(\lambda^3-\lambda^2-21\lambda+45) + \ldots$$

$$= -\lambda^3+\lambda^2+\lambda-1$$

$$\lambda^3-\lambda^2-\lambda+1 = 0$$

$$\begin{array}{r} \lambda^2 \quad -1 \\ \lambda-1 \overline{\smash{)}\lambda^3-\lambda^2-\lambda+1} \\ -\lambda^3+\lambda^2+\lambda-1 \end{array}$$

$$\lambda^3-\lambda^2-\lambda+1 = (\lambda-1)(\lambda^2-1) = (\lambda-1)^2(\lambda+1) \qquad \lambda = 1, -1$$

$\lambda = -1$

$$\begin{bmatrix} 4 & -6 & 2 \\ 2 & -4 & 2 \\ 2 & -6 & 4 \end{bmatrix} \to \begin{bmatrix} 4 & -6 & 2 \\ 2 & -4 & 2 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & -2 \\ 2 & -4 & 2 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -4 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 2 & -6 & 2 \\ 2 & -6 & 2 \\ 2 & -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The non-repeated eigenvalue $\lambda_1 = -1$ corresponds to eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The repeated eigenvalue $\lambda_2 = 1$ corresponds to the eigenvetors $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, 2 linearly independent vectors.

113 Find the eigenvalues and corresponding eigenvectors of $B = \begin{bmatrix} 8 & 1 & -7 \\ -3 & 1 & 3 \\ 4 & 1 & -3 \end{bmatrix}$

$$\det \begin{bmatrix} 8-\lambda & 1 & -7 \\ -3 & 1-\lambda & 3 \\ 4 & 1 & -3-\lambda \end{bmatrix} = (8-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} -3 & 3 \\ 4 & -3-\lambda \end{vmatrix} - 7\begin{vmatrix} -3 & 1-\lambda \\ 4 & 1 \end{vmatrix}$$

$$= -(8-\lambda)(\lambda+3)(1-\lambda) - 24 + \cancel{3\lambda}$$

$$- (\cancel{9} + \cancel{3\lambda} - \cancel{12}) \quad 3$$

$$-7(\cancel{-3}\,\cancel{-4} + 4\lambda) \quad -7$$

$$= -(8-\lambda)(-\lambda^2 - 2\lambda + 3) + 28 - 28\lambda$$

$$= (8-\lambda)(\lambda^2 + 2\lambda - 3) + 28 - 28\lambda$$

$$= -\lambda^3 + 6\lambda^2 + 19\lambda - 24 + 28 - 28\lambda$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\begin{array}{r} \lambda^2 - 5\lambda + 4 \\ \lambda - 1 \,\big)\, \overline{\lambda^3 - 6\lambda^2 + 9\lambda - 4} \\ -\lambda^3 + \lambda^2 \quad -5\lambda + 4 \\ +5\lambda^2 - 4\lambda \end{array}$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = (\lambda-1)(\lambda^2 - 5\lambda + 4)$$
$$= (\lambda-1)(\lambda-4)(\lambda-1)$$
$$= (\lambda-1)^2(\lambda-4)$$

$\lambda = 4$

$$\begin{bmatrix} 4 & 1 & -7 \\ -3 & -3 & 3 \\ 4 & 1 & -7 \end{bmatrix} \to \begin{bmatrix} 12 & 3 & -21 \\ -12 & -12 & 12 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 12 & 3 & -21 \\ 0 & -9 & -9 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 12 & 3 & -21 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 12 & 0 & -24 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 7 & 1 & -7 \\ -3 & 0 & 3 \\ 4 & 1 & -4 \end{bmatrix} \xrightarrow{①-③} \begin{bmatrix} 3 & 0 & -3 \\ -3 & 0 & 3 \\ 4 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 4 & 1 & -4 \\ 3 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 12 & 3 & -12 \\ 12 & 0 & -12 \\ & & \end{bmatrix} \to \begin{bmatrix} 12 & 3 & -12 \\ 0 & -3 & 0 \\ & & \end{bmatrix} \to \begin{bmatrix} 12 & 0 & -12 \\ 0 & -3 & 0 \\ & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The non-repeated eigenvalue $\lambda_1 = 4$ corresponds to evec $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

The repeated eigenvalue $\lambda_2 = 1$ corresponds to eve $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Note: In this case, only one eigenvector

11.4) $A = \begin{bmatrix} -7 & -10 \\ 1 & -1 \end{bmatrix}$ The eigenvalue $-4+i$ assct'd with evec $\begin{bmatrix} -3-\frac{1}{i} \\ 1 \end{bmatrix}$

The eigenvalue $-4-i$ assct'd with evec $\begin{bmatrix} -3+\frac{1}{i} \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -3-i & -10 \\ 1 & 3-i \end{bmatrix} \rightarrow \begin{bmatrix} -i & -1-3i \\ i & 3i+1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ i & 3i+1 \end{bmatrix} \rightarrow \begin{bmatrix} i & 3i+1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-(3i+1)s}{i} \\ s \end{bmatrix} = \begin{bmatrix} \frac{-(3i^2+i)}{1} s \\ s \end{bmatrix} = s \begin{bmatrix} -3+i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -7 & -10 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -3+i \\ 1 \end{bmatrix} = (-4+i) \begin{bmatrix} -3+i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -7 & -10 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -3+\frac{1}{i} \\ 1 \end{bmatrix} = (-4+i) \begin{bmatrix} -3-i \\ 1 \end{bmatrix}$$

11.5) $B = \begin{bmatrix} -4 & -10 \\ 4 & 8 \end{bmatrix}$ The eigvalue assct'd with evec $\begin{bmatrix} 1-2i \\ -1+i \end{bmatrix}$ is $2+2i$

The eigvalue assct'd with evec $\begin{bmatrix} 1+2i \\ -1-i \end{bmatrix}$ is $2-2i$

$$\begin{bmatrix} -4 & -10 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1-2i \\ -1+i \end{bmatrix} = \begin{bmatrix} 6-2i \\ -4 \end{bmatrix} = \lambda \begin{bmatrix} 1-2i \\ -1+i \end{bmatrix}$$

$$-4 = \lambda(-1+i)$$

$$\lambda = \frac{-4}{i-1}\left(\frac{i+1}{i+1}\right) = \frac{-4-4i}{-2} = 2+2i$$

$$\lambda = 2-2i$$

118) $A = \begin{bmatrix} 3 & 2 & 4 \\ -6 & -3 & -5 \\ -4 & -2 & -5 \end{bmatrix}$

Matlab

egval $-2+i$ asso'd with egvec $\langle 1, \frac{i-1}{2}, -1 \rangle$

$-2-i$ $\langle 1, \frac{-i-1}{2}, -1 \rangle$

-1 $\langle 1, 2, -2 \rangle$

$$\begin{bmatrix} 5-i & 2 & 4 \\ -6 & -1-i & -5 \\ -4 & -2 & -3-i \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$\lambda=0$ $\langle -2, 1, 0\rangle$
$\lambda=3+i$ $\langle 1, i+1, i+1\rangle$
$\lambda=3-i$ $\langle 1, i-1, i-1\rangle$

11.a) Find the eigvals and eigvecs of $\begin{bmatrix} 2 & 4 & -3 \\ -2 & -4 & 8 \\ -2 & -4 & 8 \end{bmatrix}$

$$(2-\lambda)\begin{vmatrix} (-4-\lambda) & 8 \\ -4 & (8-\lambda) \end{vmatrix} - 4\begin{vmatrix} -2 & 8 \\ -2 & 8-\lambda \end{vmatrix} - 3\begin{vmatrix} -2 & -4-\lambda \\ -2 & -4 \end{vmatrix}$$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda\right] - 4(2\lambda) - 3(\cancel{8} - 2\lambda)$$

$$= -\lambda^3 + 6\lambda^2 - 16\lambda + 6\lambda$$

$$= -\lambda^3 + 6\lambda^2 - 10\lambda$$

$$\lambda^3 - 6\lambda^2 + 10\lambda = \lambda(\lambda^2 - 6 + 10)$$

$$\frac{6 \pm \sqrt{36-40}}{2} = 3 \pm \frac{2i}{2} = 3 \pm i$$

$$\lambda = 0,\ 3+i,\ 3-i$$

$\lambda=0$

$$\begin{bmatrix} 2 & 4 & -3 \\ -2 & -4 & 8 \\ -2 & -4 & 8 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 5 \\ 2 & 4 & -8 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ & & \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -0 \\ 0 & 0 & 1 \\ & & \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda=3+i$

$$\begin{bmatrix} -1-i & 4 & -3 \\ -2 & -7-i & 8 \\ -2 & -4 & 8-i \end{bmatrix} \begin{bmatrix} -1-i & 4 & -3 \\ 0 & -3-i & i \\ 2 & 4 & i-8 \end{bmatrix} \begin{bmatrix} -i & 6 & -7+i/2 \\ i & 2i & -\frac{1}{2}-4i \\ 0 & -3-i & i \end{bmatrix}$$

$$\begin{bmatrix} 0 & 6+2i & -7.5-3.5i \\ i & 2i & -\frac{1}{2}-4i \\ 0 & -3-i & i \end{bmatrix} \begin{bmatrix} 1 & 2 & \frac{-1}{2i}-4 \\ 0 & -6-2i & 2i \\ 0 & 6+2i & -7.5-3.5i \end{bmatrix} \begin{bmatrix} 1 & 2 & -4-\frac{1}{2i} \\ 0 & 0 & -7.5-1.5i \\ 0 & 6+2i & -7.5-3.5i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4-\frac{1}{2i} \\ 0 & 6+2i & -2i \\ 0 & 0 & -7.5-1.5i \end{bmatrix}$$

Eigenbasis of an $n \times n$ matrix, A

a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of eigenvectors of A

such a basis may or may not exist.

- It exists when **there are m linearly independent eigenvectors for each eigenvalue of algebraic multiplicity m**

  algebraic multiplicity: e.g. $p(\lambda) = -\lambda(\lambda-1)^3(\lambda-3)^4$ (7×7)
  eigenvalues 0, 1, 3 with algebraic multiplicities 1, 3, and 4 respectively.

- If A **has n distinct eigenvalues**, then A has an eigenbasis.

$\boxed{\text{e.g.}}$ $A = \text{Proj}_{3x=4y} = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$ 0, 1 are eigen values of A,
$\Rightarrow$ 2 distinct eigenvalues for a 2×2 matrix
$\Rightarrow$ A has an eigenbasis

$p(\lambda) = \lambda^1(\lambda-1)^1$

A has an eigenbasis $\left\{ \underset{\substack{\text{evec of eval } 0 \\ \perp L}}{\begin{bmatrix} -3 \\ 4 \end{bmatrix}}, \underset{\substack{\text{evec of eval } 1 \\ \text{on } L}}{\begin{bmatrix} 4 \\ 3 \end{bmatrix}} \right\}$. See Geometric Example.

$\boxed{\text{e.g.}}$ $B = \begin{bmatrix} 3 & -6 & -7 \\ 1 & 8 & 5 \\ -1 & -2 & 1 \end{bmatrix}$ 2, 4, 6 are eigenvalues all with alg multiplicity 1,
$\Rightarrow$ 3 distinct evals for 3×3 matrix.

$p(\lambda) = -(\lambda-2)(\lambda-4)(\lambda-6)$

B has an eigenbasis $\left\{ \overset{2}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}, \overset{4}{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}, \overset{6}{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}} \right\}$

$\boxed{\text{e.g.}}$ $C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 2 \end{bmatrix}$ evalues: -1, alg mult 2
2, alg mut 1

$p(\lambda) = -(\lambda+1)^2(\lambda-2)$

For $\lambda = 2$, $C - \lambda I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = -1$, $C - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

There is only one free variable!
Only one lin. independent evec for e.val -1.
$\Rightarrow$ # of lin. indep. evec of eval -1 < alg mult of -1

C does not have an eigenbasis (C doesn't have enough eigenvectors)

If a matrix has an eigenbasis, you can treat it as a linear transformation?

10.7) If $\vec{v_1} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ are eigvecs of a matrix A corresponding to $\lambda_1 = 5$ and $\lambda_2 = 1$ respectively, then

$$A\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$\begin{aligned} A(\vec{v_1} + \vec{v_2}) &= A\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix}\right) \\ &= A\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) + A\left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}\right) \\ &= 5\begin{bmatrix} 5 \\ -3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 25 \\ -15 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 29 \\ -11 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A(-3\vec{v_1}) &= A\left(-3\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) \\ &= -3A\begin{bmatrix} 5 \\ -3 \end{bmatrix} \\ &= -3(5)\begin{bmatrix} 5 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -75 \\ 45 \end{bmatrix} \end{aligned}$$

10.8) Given that the matrix B has eigval $\lambda_1 = 4$ corresponding to eigvec $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and eigval $\lambda_2 = 8$ corresponding to eigvec $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, find B.

Hint: Treat $Bx = T(x)$ as a linear transformation.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(2\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 5\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right)(-1) = -2\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 5\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} B\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= B\left(-2\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 5\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right) = -2B\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 5B\begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= -2(8)\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 5(4)\begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -32 \\ -80 \end{bmatrix} - \begin{bmatrix} -20 \\ -40 \end{bmatrix} = \begin{bmatrix} -12 \\ -40 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= B\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right) = B\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2B\begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2(4)\begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 16 \\ 40 \end{bmatrix} + \begin{bmatrix} -8 \\ -16 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 24 \end{bmatrix} \end{aligned}$$

$$B = \begin{bmatrix} | & | \\ B(e_1) & B(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} -12 & 8 \\ -40 & 24 \end{bmatrix}$$

11.7) Let A be the standard matrix of the lin. trans. that rotates any 2D vector $\vec{x}$ by t radians CCW about the origin.

a) Find the matrix A in terms of t.

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

b) Find the eigenvalues and corresponding eigenvectors of A if $t = \pi/3$

$\cos(\pi/3) = 1/2 \qquad \sin(\pi/3) = \sqrt{3}/2$

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1/2 - \lambda & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 - \lambda \end{bmatrix} \text{ or } \begin{bmatrix} 1-\lambda & -\sqrt{3} \\ \sqrt{3} & 1-\lambda \end{bmatrix}$$

can do b/c making det $=0$?

$(1/2 - \lambda)^2 + 3/4 \qquad\qquad (1-\lambda)^2 + 3$

$\lambda^2 - \lambda + 1 = 0 \qquad\qquad \lambda^2 - 2\lambda + 4 = 0$ Nope.

roots: $\dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}\sqrt{-1}}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$

$\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} i & -\sqrt{3}/2 \\ \sqrt{3}/2 & -\frac{\sqrt{3}}{2} i \end{bmatrix} \to \begin{bmatrix} 0 & -\sqrt{3}/2 - \cancel{\frac{\sqrt{3}}{2} i^2} \\ \sqrt{3}/2 & -\frac{\sqrt{3}}{2} i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}, \; s\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Complex conj trick

$\lambda = \frac{1}{2} - \frac{\sqrt{3}}{2} i$, eigvec $\begin{bmatrix} -i \\ 1 \end{bmatrix}$