Complex Numbers

Numbers:	Natural	0, 1, 2, 3... (some ppl say no 0)
	Integers	$0, \pm 1, \pm 2, \pm 3$
	Rational	$2/5, -1/7, 9/2$
	Real	$\sqrt{2}, \pi, e$
	Complex	$i, 2+3i$

i is a number s.t. $i^2 = -1$. As a consequence, $x^2+1=0$, which has no real sol'n, has sol'n $x = i$.

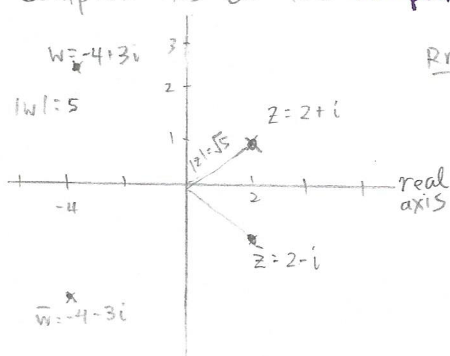
A **complex number** is of the form $z = a + bi$ where a, b are real #s.
eg. $1+2i$, $4i$, $e+\pi i$ This is rectangular form.

'a' is the real part of z 'b' is the imaginary part of z.

modulus / length: $|z| = \sqrt{a^2+b^2}$ e.g. $z = 2+3i$ $|z| = \sqrt{2^2+3^2}$
conjugate: $\bar{z} = a - bi$ $\bar{z} = 2-3i$

Geometric Representation

- Real #s on the real # line
- Complex #s on the **complex plane**.



Rmks
$\bar{z}$ is the mirror img of z wrt the real axis
$|z|$ is the distance b/w 0 and z.

Adding complex #s on a complex plane is like vector addition

Arithmetic

do $+, -, \times, \div$

- $(1+2i)+(4-3i) = 5+i$
- $(1+2i)-(4-3i) = -3+5i$
- $(1+2i)(4-3i) = 4+5i-6i^2 = 4+5i-6(-1) = 10+5i$
- $(1+i)(1-i) = 1-i^2 = 1-(-1) = 2$
- div later

Properties

1. $z\bar{z} = |z|^2$
   $(a+bi)(a-bi) = a^2-(bi)^2 = a^2-(-b)^2 = a^2+b^2$
2. Real part of $z = \frac{1}{2}(z+\bar{z})$
   Imag part of $z = \frac{1}{2i}(z-\bar{z})$
3. $\overline{z^n} = \bar{z}^n$ and $|z^n| = |z|^n$
4. $\overline{zw} = \bar{z}\,\bar{w}$ z, w both complex #s
   $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$
5. $|zw| = |z||w|$
   $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

Express the following in rectangular form $a + bi$.

(1) $i^4 = (i^2)(i^2) = (-1)(-1) = 1$

(2) $\frac{4-2i}{1-i} = \left(\frac{4-2i}{1-i}\right)\left(\frac{1+i}{1+i}\right) = \frac{4+2i-2i^2}{1-i^2} = \frac{6+2i}{2} = 3+i$

Rmk: Check: $(1-i)(3+i) = 3-2i-i^2 = 4-2i$ ✓

(3) $\frac{2+i}{4-3i} = \left(\frac{2+i}{4-3i}\right)\left(\frac{4+3i}{4+3i}\right) = \frac{8+10i+3i^2}{16-(-9)} = \frac{5+10i}{25} = \frac{1}{5} + \frac{2}{5}i$

(4) $\left|\left(\frac{3+i}{1-i}\right)^{10}\right|$ ^10 and then find modulus : modulus is too much work.

$= \left(\left|\frac{3+i}{1-i}\right|\right)^{10}$ $\quad |z^n| = |z|^n$

$= \left(\frac{|3+i|}{|1-i|}\right)^{10}$ $\quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

$= \left(\frac{\sqrt{10}}{\sqrt{2}}\right)^{10} = \sqrt{5}^{10} = (5^{1/2})^{10}$

$= 5^5$

Q2 Calculate the following if $a = -5+i$ and $b = 1-4i$

a) $a + b = -4-3i$

b) $2a - 4b = (-10+2i)-(4-16i) = -14+18i$

c) $ab = (-5+i)(1-4i) = -5+20i+i-4i^2 = -1+21i$

d) $\bar{a} = -5-i$

e) $|b| = \sqrt{(1^2+4^2)} = \sqrt{17}$

Q3 Write the following in the form $a + bi$

a) $\frac{-3-i}{3-2i}\left(\frac{3+2i}{3+2i}\right) = \frac{-9-9i-2i^2}{9+4} = \frac{-7-9i}{13} = -\frac{7}{13} - \frac{9}{13}i$

b) $\frac{-3}{4i} + \frac{-2}{5i} = \frac{-15-8}{20i} = \frac{-23}{20i}\left(\frac{20i}{20i}\right) = \frac{\cancel{-}460i}{\cancel{-}400} = \frac{23}{20}i$

c) $(4i)^3 = 4^3 i^3 = 4^3 i^2 i = -4^3 i$

1.4 Let $z = 3+3i$. Calculate:

a) $z^2 + 2z + 1 = (z+1)(z+1) = (4+3i)(4+3i)$

$= 16 + 24i + 9i^2$

$= 7 + 24i$

b) $z^2 + iz - (2+i)$

$= z(z+i) - (2+i)$

$= (3+3i)(3+4i) - (2+i)$

$= (9 + 21i + 12i^2) - (2+i)$

$= (-3+21i) - (2+i)$

$= -5 + 20i$

or $(9+9i^2+18i) + (3i+3i^2) - 2 - i$

$= 12i^2 + 20i + 7$

$= -5 + 20i$

c) $\dfrac{(z-4)^2}{z+i} = \dfrac{(-1+3i)^2}{3+4i}$

$= \dfrac{1-6i+9i^2}{3+4i}$

$= \dfrac{-8-6i}{3+4i}\left(\dfrac{3-4i}{3-4i}\right) = \dfrac{-24+14i+24i^2}{25}$

$= \dfrac{-48+14i}{25} = \dfrac{-48}{25} + \dfrac{14}{25}i$

Exponent of a Complex Number

Let $z = a+bi$ (a,b real) be a complex number.

We define $$e^z = e^{a+bi} = e^a(\cos b + i \sin b) \quad {}^{*}b \text{ in radians}$$

e.g. $e^i = e^0(\cos 1 + (i)\sin 1)$

e.g. $e^{\phi i} = e^0(\cos\phi + i\sin\phi)$ Rmk $|e^{i\phi}| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$

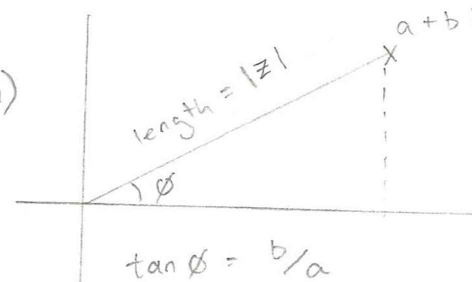
e.g. $e^{2+\pi/4\, i} = e^2(\cos(\pi/4) + i\sin(\pi/4))$
$= e^2/\sqrt{2} + e^2/\sqrt{2}\, i$

e.g. $e^{\pi i} = e^0(\cos(\pi) + \sin(\pi)i) = -1$

Prop: If z, w are complex numbers:
$e^z \cdot e^w = e^{z+w}$
$e^z / e^w = e^{z-w}$
$1/e^z = e^{-z}$

Polar Form of a Complex Number

$z = a + bi$ (rectangular form)
$= |z|e^{i\phi} = |z|(\cos\phi + i\sin\phi)$ (polar form)



$\tan\phi = b/a$

$\phi = \arctan(b/a)$ if in quadrant 1.

Rmk: $z_1 = |z_1|e^{i\phi_1}$ $z_2 = |z_2|e^{i\phi_2}$
$z_1 z_2 = |z_1||z_2|e^{i(\phi_1+\phi_2)}$

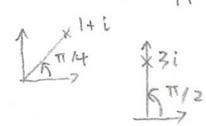
Rmk: $+, -,$ easier in rectangular form
$\times, \div,$ easier in polar form.

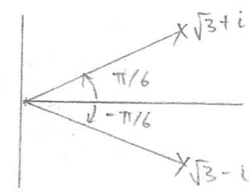
$z_1 = 1+i$ $|z_1| = \sqrt{1^2+1^2} = \sqrt{2}$ $z_1 = \sqrt{2}e^{i\pi/4}$

$z_2 = 3i$ $|z_2| = \sqrt{9} = 3$ $z_2 = 3e^{i\pi/2}$

$z_3 = \sqrt{3}+i$ $|z_3| = \sqrt{\sqrt{3}^2+1^2} = \sqrt{4} = 2$ $z_3 = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$ $e^{i\phi} = \cos\phi + i\sin\phi$
$= 2e^{i\pi/6}$

$z_4 = \sqrt{3}-i$ $|z_4| = 2$ $z_4 = 2e^{-\pi i/6}$

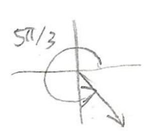




$z_1 z_2 = (\sqrt{2}e^{\pi/4\, i})(3e^{\pi/2\, i}) = 3\sqrt{2}\,e^{3\pi/4\, i}$ (polar form)
$= 3\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$
$= 3\sqrt{2}(-1/\sqrt{2} + 1/\sqrt{2}\, i) = -3+3i$ (rectangular form)

$z_1/z_2 = (\sqrt{2}e^{\pi/4 i})/(3e^{\pi/2 i}) = \sqrt{2}/3\, e^{-\pi/4\, i}$ (polar form)

$z_3^{10} = (2e^{\pi/6 i})^{10} = 2^{10}e^{\frac{5\pi}{3}i}$ (polar form)
$= 2^{10}(\cos(5\pi/3) + \sin(5\pi/3)i)$
$= 2^{10}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= 512 - 512\sqrt{3}i$



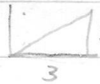
9.6 Write the following in the polar form $re^{i\phi}$, $r \geq 0$ and $0 \leq \phi < 2\pi$

a) $-1/9 + 0i$, length $= 1/9$. $z = 1/9\, e^{\pi i}$

b) $5+5i$ length $= \sqrt{50} = 5\sqrt{2}$ $\pi/4$ $z = 5\sqrt{2}\, e^{i\pi/4}$

c) $-5+5i$ $\phi = 3\pi/4$ $z = 5\sqrt{2}\, e^{\frac{3\pi}{4} i}$

d) $3+2i$ length $= \sqrt{13}$

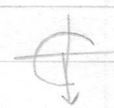


$z = \sqrt{13}\, e^{\arctan(2/3) i}$

9.7 Write each of the given numbers in the form $a+bi$

a) $4e^{3i\pi/4} = 4(\cos(3\pi/4) + \sin(3\pi/4) i)$
$= 4(-1/\sqrt{2} + 1/\sqrt{2}\, i)$
$= -4/\sqrt{2} + 4/\sqrt{2}\, i$

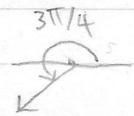
b) $e^{(5+3i\pi)} e^{(3 + \frac{i\pi}{2})} = e^{8 + \frac{7\pi}{2} i}$
$= e^8 \left(\cos\left(\frac{7\pi}{2}\right) + i \sin\left(\frac{7\pi}{2}\right)\right)$
$= e^8 (0 - i) = -e^8 i$



c) $\dfrac{6e^{(5+3i\pi)}}{5e^{(3+i\pi/2)}} = \frac{6}{5} e^{(2 + \frac{5\pi i}{2})} = \frac{6}{5}\left(e^2 \left(\cancel{\cos(\pi/2)} + \sin(\pi/2) i\right)\right)$
$= \frac{6}{5} e^2 (i)$

9.8 a) $(e^{-2+5i})^{-2} = e^{4-10i} = e^4(\cos(-10) + \sin(-10) i)$
$= e^4 \cos(-10) + e^4 \sin(-10) i$

b) $(-1-i)^{22}$ $z = -1 - i = \sqrt{2} e^{i \arctan(1)} = \sqrt{2} e^{3\pi/4\, i}$



$= (\sqrt{2} e^{3\pi/4\, i})^{22}$
$= 2^{11} e^{66\pi/4\, i} = 2^{11} e^{3\pi/4\, i} = 2^{11}(-1 i)$

Factorization of Polynomials

$x^2-4x+3=0 \Rightarrow (x-3)(x-1)=0 \quad x=3$ or 1

$x^2-4x+5=0$

! cannot be factorized as $(x-a)(x-b)$ for real a, b

a) $= (x-2)^2+1=0$ (completing the square)

$(x-2)^2 = -1 = i^2$

$x-2 = \pm i$

$x = 2 \pm i$

b) $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ (quadratic formula)

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2-4(1)(5)}}{2(1)}$

$= \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4}{2} \pm \dfrac{2\sqrt{-1}}{2} = 2 \pm i$

9.9 **Find all solutions to $z^2+3z+5=0$ and express them as $a+bi$**

$(z+3/2)^2 + 11/4 = 0 \qquad 5 - 9/4 = \frac{20-9}{4} = 11/4$

$(z+3/2)^2 = -11/4 = i^2(11/4)$

$z+3/2 = (\pm i\sqrt{11/4})$

$z = -3/2 + i\sqrt{11/4},\ -3/2 - i\sqrt{11/4}$

9.10 **Find all the values of $(-27)^{1/3}$**

Express (-27) in polar form. *

$-27 = 27e^{(\pi + 2k\pi)i}$

$(-27)^{1/3} = 3e^{(\pi/3 + \frac{2}{3}k\pi)i}$

$k=0 \quad 3e^{\frac{\pi i}{3}} = 3(\cos(\frac{\pi}{3}) + \sin(\frac{\pi}{3})i) = 3(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$

$k=1 \quad 3e^{\pi i} = 3(\cos(\pi) + \sin(\pi)i) = 3(-1+0)$

$k=2 \quad 3e^{\frac{5}{3}\pi i} = 3(\cos(\frac{5}{3}\pi) + \sin(\frac{5}{3}\pi)i) = 3(\frac{1}{2} - \frac{\sqrt{3}}{2}i)$

$k=3 \quad 3e^{\frac{7}{3}\pi i} = 3e^{\frac{\pi}{3}i}$

9,5 $\left|\frac{2+2i}{-2-i}\right| = \frac{|2+2i|}{|-2-i|} = \frac{\sqrt{2(4)}}{\sqrt{4+1}} = \frac{2\sqrt{2}}{\sqrt{5}}$

$$\left|\overline{(1+i)}(2-3i)(2-2i)\right| = \left|\overline{(1+i)}\right|\left|(2-3i)\right|\left|(2-2i)\right|$$
$$= \sqrt{2}\sqrt{13}\sqrt{8} = 4\sqrt{13}$$

$$\left|\frac{i(2-i)^3}{(1-i)^2}\right| = \left|\frac{i\left[(\cancel{i^2}^{\,3}-4i+\cancel{4})(2-i)\right]}{1-2i+i^2}\right| = \left|\frac{\cancel{i}(3-4i)(2-i)}{-2\cancel{i}}\right|$$
$$= \left|\frac{6-11i+4i^2}{-2}\right| = \left|\frac{2-11i}{-2}\right| = \frac{\sqrt{4+121}}{2} = \frac{5\sqrt{5}}{2}$$

$$\left|\frac{(\pi+i)^{52}}{(\pi-i)^{52}}\right| = \left|\frac{(\pi+i)}{(\pi-i)}\right|^{52} = \left(\frac{|\pi+i|}{|\pi-i|}\right)^{52} = \left(\frac{\sqrt{\pi^2+1}}{\sqrt{\pi^2+1}}\right)^{52} = 1$$