Minor

Given an $n \times n$ matrix A, the **minor** is defined as the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i-th row and j-th column. **M_{ij}**

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$ $M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$

Determinants

1×1 $A = [a_{11}]$ $\det([a_{11}]) = a_{11}$

$n \times n$ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \vdots & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \cdots & & & a_{nn} \end{bmatrix}$

$\det(A) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} + \dots + (-1)^{1+n} a_{1n} M_{1n}$

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+n} a_{ij} M_{ij}$$

expansion along row 1

2×2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a_{11} = a$, $M_{11} = |d| = d$ $a_{12} = b$, $M_{12} = |c| = c$

$\det(A) = ad - bc$ ✓

3×3 $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$\det(A) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$
$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad ✓$$

Expansion along different rows:

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$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for a fixed i, expansion along the i^{th} row

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for a fixed j, expansion along the j^{th} column

fact: $\det(A) = \det(A^T)$

columns of A are rows of A^T

e.g.
$$\begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} & j=2 & \\ \end{array}$$
$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \sum_{i=1}^{3} (-1)^{i+2} a_{i2} M_{i2}$$
$$= (-1)(1)\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 0 = -3$$

$i=3$
$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \sum_{j=1}^{3} (-1)^{3+j} a_{3j} M_{3j}$$
$$= (1)(1)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - 0 + (1)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -3$$

Rmk About Sign for $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix}$$

e.g.

$i=2$
$$\begin{vmatrix} 0 & 3 & -1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 4 & 5 & 2 & 6 \\ 1 & 5 & -1 & 0 & -2 \\ 0 & 6 & 1 & 0 & 2 \end{vmatrix} = \sum_{j=1}^{5} (-1)^{i+j} a_{2j} M_{2j} = \begin{vmatrix} 0 & -1 & 0 & -3 \\ 3 & 5 & 2 & 6 \\ 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 2 \end{vmatrix}$$

$j=3$
$$= \sum_{i=1}^{4} (-1)^{i+j} a_{i3} M_{i3} = (-1)(2)\begin{vmatrix} 0 & -1 & -3 \\ 1 & -1 & -2 \\ 0 & 1 & 2 \end{vmatrix}$$
$$= (-2)\left(-1\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix}\right)$$
$$= 2\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix}$$
$$= 2$$

Determinants, useful properties

① $\det\left(\begin{array}{c}\text{upper/lower}\\ \text{triangular matrix}\end{array}\right)$ = product of diagonal entries

e.g. $\det\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix} = 1\cdot4\cdot6 = 24$ $\det\begin{bmatrix}1&0&0\\0&2&0\\10^7&9&0\end{bmatrix} = 2\cdot1\cdot0 = 0$ $\det(I_n) = \det\begin{bmatrix}1&&0\\&\ddots&\\0&&1\end{bmatrix} = (1)^n = 1$

② $\det(A) = 0 \iff A$ is not invertible

③ $A, B : n\times n$ $\det(AB) = \overset{\text{numbers}}{(\det A)(\det B)}$ AB invertible, both A, B are invertible

④ $A: n\times n$ $\det(A^T) = \det(A)$

* ⑤ $A: n\times n$. B obtained from a by

- ⓐ swapping rows $\det(B) = -\det(A)$
- ⓑ mult row by a constant k $\det(B) = k\det(A)$
- ⓒ adding a multiple of a row to another. $\det(B) = \det(A)$
- ⓐ' ⓑ' ⓒ' similarly for column operations

e.g. $\begin{vmatrix}1&2&3\\1&2&1\\2&3&0\end{vmatrix} \underset{R_2-R_1}{\overset{(5c)}{=}} \begin{vmatrix}1&2&3\\0&0&-2\\2&3&0\end{vmatrix} \overset{(5c)}{=} \begin{vmatrix}1&2&3\\0&0&-2\\0&-1&-6\end{vmatrix} \overset{(5a)}{=} -\begin{vmatrix}1&2&3\\0&-1&-6\\0&0&-2\end{vmatrix} = (1)(-1)(-2) = 2$

e.g. $\begin{vmatrix}2&2&2&2\\3&3&3&3\\4&5&6&7\\8&9&10&11\end{vmatrix} \overset{(5b)}{=} (2)(3)\begin{vmatrix}1&1&1&1\\1&1&1&1\\4&5&6&7\\8&9&10&11\end{vmatrix} = 0$

⑥ If a matrix has 2 identical rows or columns, then its determinant = 0

e.g. $\begin{vmatrix}1&1&1&1\\1&1&1&1\\4&5&6&7\\8&9&10&11\end{vmatrix} \overset{\updownarrow\,(5a)}{=} -\begin{vmatrix}1&1&1&1\\1&1&1&1\\4&5&6&7\\8&9&10&11\end{vmatrix}$ det is 0.

⑦ Any matrix with a 0 row/column has det = 0, (see xpansion along diff rows/cols)

⑧ $A: n\times n$, $\det(A^n) = (\det A)^n$

if A is invertible, $\det(A^{-1}) = (\det A)^{-1} = \frac{1}{\det A}$

$\Rightarrow$ the determinant is a #

De True or False, A, B are $n \times n$

① $\det(AB) = \det(BA)$ True

In general, $AB \neq BA$, but $\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$

② $\det(A+B) = \det A + \det B$ False

Counterexample $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$, $\det(A) + \det(B) = 0$

③ $(A+B)^2 = A^2 + 2AB + B^2$ False.

④ $\det A = 4 \Rightarrow \det(3A) = 12$ False

$A: \begin{bmatrix} - a_1 - \\ - a_2 - \\ - a_n - \end{bmatrix}$ $3A = \begin{bmatrix} - 3a_1 - \\ - 3a_2 - \\ - 3a_n - \end{bmatrix}$

$$\det(3A) = 3 \det \begin{bmatrix} - a_1 - \\ - 3a_2 - \\ - 3a_n - \end{bmatrix}$$

$$= 3^n \det \begin{bmatrix} - a_1 - \\ - a_2 - \\ - a_n - \end{bmatrix}$$

$$= 3^n \det A$$

$\det(3A) = 3^n \det A$

⑤ $A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}$ $B = \begin{bmatrix} | & | & | & & | \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ | & | & | & & | \end{bmatrix}$ False.

B obtained by 'moving' last column to the first one.

Then $\det B = -\det A$

$n=3$

$$\det B = \begin{vmatrix} | & | & | \\ a_3 & a_1 & a_2 \\ | & | & | \end{vmatrix} = -\begin{vmatrix} | & | & | \\ a_1 & a_3 & a_2 \\ | & | & | \end{vmatrix} = \begin{vmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{vmatrix} = \det A$$

$n=4$

$$\det B = \begin{vmatrix} | & | & | & | \\ a_4 & a_1 & a_2 & a_3 \\ | & | & | & | \end{vmatrix} = -\begin{vmatrix} | & | & | & | \\ a_1 & a_4 & a_2 & a_3 \\ | & | & | & | \end{vmatrix} = \begin{vmatrix} | & | & | & | \\ a_1 & a_2 & a_4 & a_3 \\ | & | & | & | \end{vmatrix} = -\begin{vmatrix} | & | & | & | \\ a_1 & a_2 & a_3 & a_4 \\ | & | & | & | \end{vmatrix} = -\det A$$

$$\det B = (-1)^{n-1} \det A$$

$$= \begin{cases} \det A & \text{if } n = \text{odd} \\ -\det A & \text{if } n = \text{even} \end{cases}$$

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

A is invertible $\iff$ $v_1, \ldots v_n$ form a basis of $\mathbb{R}^n$

$rref(A) = I_n$

basis: n vectors in $\mathbb{R}^n$, linearly independent

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \iff c_1 \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} + c_2 \begin{bmatrix} | \\ v_2 \\ | \end{bmatrix} + \cdots + c_n \begin{bmatrix} v_n \end{bmatrix} = \vec{0}$$

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

under 'properties'

$A\vec{x} = \vec{b}$, A is invertible.

always has unique solution for any $\vec{b}$.

Non-square [ ] can't be invertible

If $A: m \times n$, $C: n \times m$, $m \neq n$

$C = A^{-1}$

$AB = I_m$ < one is impossible > $CA = I_n$

$(CA)B = CI_m$

$(I_n)B = CI_m$

$B = C$

Determinants

8.16) A, B are 3×3 $\quad \det(A) = -1 \quad \det(B) = -8$

a) $\det(AB) = (\det A)(\det B) = 8$

b) $\det(-2A) = (-2)^n(\det A) = (-2)^3(-1) = 8$

c) $\det(A^T) = \det(A) = -1$

d) $\det(B^{-1}) = \frac{1}{\det B} = -1/8$

e) $\det(B^2) = (\det B)^2 = 64$

8.17) Given $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -4$

a) $\det \begin{bmatrix} a & b & c \\ d & e & f \\ 7g & 7h & 7i \end{bmatrix} = 7 \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -28$

b) $\det \begin{bmatrix} a & b & c \\ d-7g & e-7h & f-7i \\ g & h & i \end{bmatrix} = -4$, adding mult row $\det(A) = \det(B)$

c) $\det \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = 4$

d) $\det \begin{bmatrix} -5d+4a & -5e+4b & -5f+4c \\ 6g & 6h & 6i \\ a & b & c \end{bmatrix}$ 2 flips, same sign as -4.
$(-4)(-5)(6) = 120$

8.15) Find k such that $\begin{bmatrix} -1 & -4 & 3 \\ 0 & -4 & -3 \\ 6+k & 12 & -3 \end{bmatrix}$ is invertible.

invertible if $\det \neq 0$.

$-1 \begin{vmatrix} -4 & -3 \\ 12 & -3 \end{vmatrix} = -1(48)$

$\det = 0$ if $(6+k)\begin{vmatrix} -4 & 3 \\ -4 & -3 \end{vmatrix} = 48$

$(6+k)(24) = 48$

$k = -4$

The matrix is invertible when $k \neq -4$, $k = -4$ makes $\det = 0$.

8.12) $\det \begin{bmatrix} 2 & 1 & 8 & -1 \\ 0 & -7 & 8 & 4 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 2(-7)(-7)(4) = 392$

8.13) $\det \begin{bmatrix} -6 & -2 & -5 \\ -1 & -1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 0 & 4 & 25 \\ -1 & -1 & -5 \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & -1 & -5 \\ 0 & 4 & 25 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 4 & 25 \\ 0 & 0 & 1 \end{vmatrix} = 4$

8.14) $\det \begin{bmatrix} 3 & 0 & 0 & -2 \\ -3 & 0 & -3 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -3 & 2 & 0 \end{bmatrix} = -3 \begin{vmatrix} 3 & 0 & 0 & -2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -3 & 2 & 0 \end{vmatrix}$

$= -3 \left[(-1) \begin{vmatrix} 0 & 0 & -2 \\ 1 & 0 & -2 \\ -3 & 2 & 0 \end{vmatrix} - 1 \left((-1)(-3) \begin{vmatrix} 3 & -2 \\ 0 & -2 \end{vmatrix} \right) \right]$

$= -3 \left[(-1)(-2)(2) - 3(-6) \right]$

$= -3 \left[4 + 18 \right]$

$= -66$