Inverse

e.g. $2x = 4$ ($\times 2$, $\times \frac{1}{2}$)

$x = 2$

'Multiplying by 2 and $\frac{1}{2}$ are inverse to each other'

MATRIX INVERSES: Do similarly to solve

$$Ax = b \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad A: m \times n \quad b: m \times 1 \qquad Ax = (m \times n)(n \times 1)$$

Def'n: Suppose A is $n \times n$. A is said to be **invertible** if exists an $n \times n$ matrix B such that

$$BA = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

In this case, we say that B is the **inverse** of A. $A^{-1} = B$.

$\boxed{\text{e.g.}}$ $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

By definition, A is invertible and $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$\begin{cases} 2x - y = 3 \\ -x + y = -1 \end{cases} \iff \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$BA\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$I_2\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad x = 2,\ y = 1$$

$A\vec{x} = b$

$(BA)\vec{x} = (B)b$

$I_n\vec{x} = (B)b$

$\vec{x} = Bb$

$\boxed{\text{Rmk}}$ If A, B are $n \times n$, $BA = I_n$, then $AB = I_n$. If $A^{-1} = B$, $B^{-1} = A$

$\boxed{\text{Rmk}}$ Not all square matrices have inverse. $[0]$ is not an invertible 1×1 matrix.

Propertys

Let A be an $\underline{n \times n}$ matrix. Then the following are equivalent:

1. A is invertible
2. Each diagonal entry of a **ref** of A is non-zero
3. rref $A = I_n$
4. $A\vec{x} = \vec{b}$ always has a solution
5. $A\vec{x} = \vec{0}$ has only one sol'n, $\vec{x} = \vec{0}$ (see linear independence)
6. $\det A \neq 0$ - discuss det for $n \times n$, $n > 3$ matrices later
7. rank $A = n$
8. Columns (rows) of A are linearly independent vectors in $\mathbb{R}^n$.

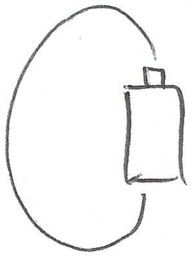
if $m \neq n$ either one impossible

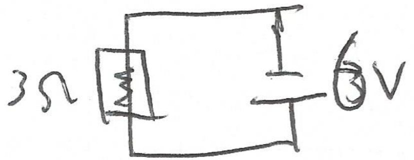
$AB = I_m \quad \swarrow \quad \searrow \quad CA = I_n$

$CAB = C I_m$ $\qquad A \quad m \times n$

$B = C I_m$ $\qquad C \quad n \times m$

$= C$





$P = I^2 R$

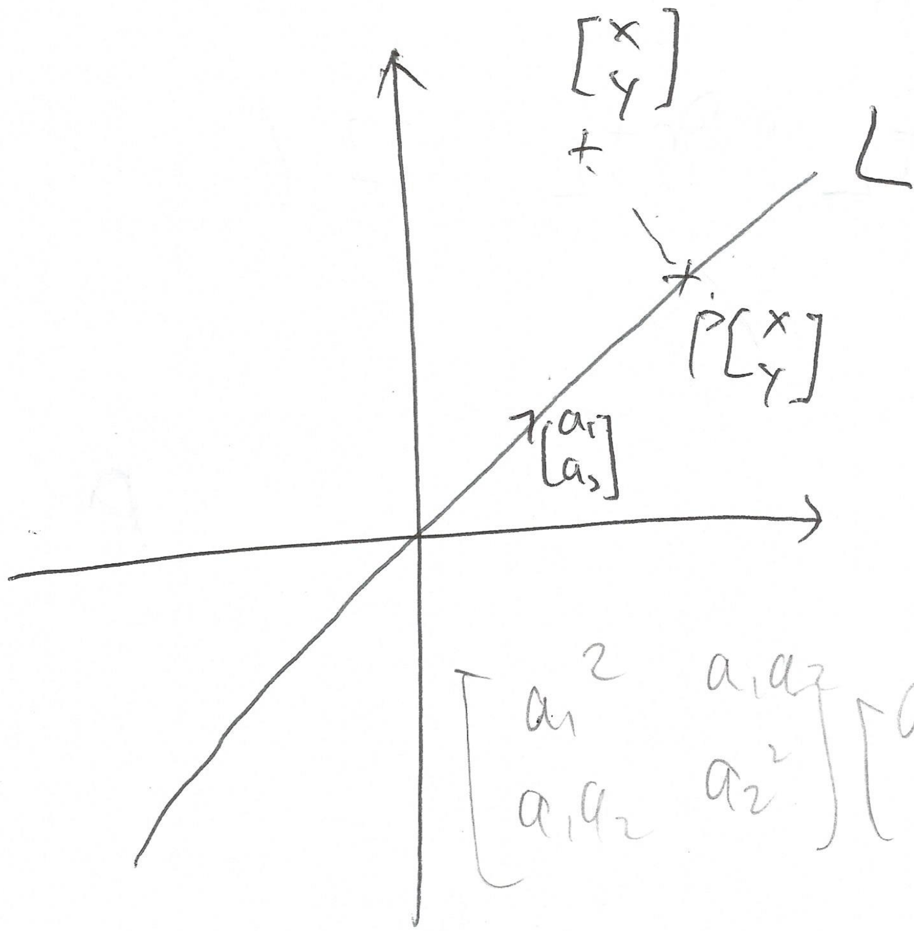
$Id : \mathbb{R}^2 \to \mathbb{R}^2 \quad Id(\vec{x}) = \vec{x} \leftrightarrow I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$Ref_L \circ Ref_L = Id$

$Q^2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ for any $\begin{bmatrix} x \\ y \end{bmatrix}$

$$Proj_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b}$$

$$= \begin{bmatrix} \frac{x+3y}{10} \\ \frac{3x+9y}{10} \end{bmatrix}$$



$$\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} a_1^2 & \end{bmatrix}$$

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e.g.

$$\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^4 + a_1^2a_2^2 & \\ & \end{bmatrix} = \begin{bmatrix} a_1^2(a_1^2+a_2^2) & \\ & \end{bmatrix}$$

$$= \begin{bmatrix} a_1^2 & \\ & \end{bmatrix}$$

Finding the Inverse

Given A: $n \times n$ matrix,

- If $\text{ref}(A)$ has a non-0 entry in its diagonal, A is not invertible
- Else A is invertible and $\text{rref}(A) = I_n$

Method to Find A^{-1}:

$$[A \mid I_n] \xrightarrow[\text{ops}]{\text{row}} [\text{rref}(A) \mid B]$$

$$B = A^{-1} \text{ if } \text{rref}(A) = I_n$$

e.g. Determine whether these are invertible. If they are find the inverse.

A) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ already in ref, non-0 diagonal, invertible ✓

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{(1)-2(2)} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

c) $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{(1)\leftrightarrow(2)} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{(2)+(1)} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

0 in diagonal

C is not invertible

D) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & 1 & -3 \end{array}\right]$$

→ ref(A) no non-0 diagonal, invertible ✓

rref D: I_3 $\qquad D^{-1}$

$$D^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 4 \\ 0 & 1 & -3 \end{bmatrix}$$

Why $[A \mid I_n] \to [\text{rref}\, A \mid A^{-1}]$ computes inverse?

A: $m \times n$ $\qquad B$: $n \times k$ $\qquad B$ has $n \times 1$ cols in $\mathbb{R}^n$ $\quad b_1, b_2, \ldots b_k$

$$AB = A\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_k \\ | & | & & | \end{bmatrix} \quad (*)$$

Ab_i are $m \times 1$

If $m = n = k$, A, B are square matrix.

$AB = I_n \iff Ab_1 = e_1, \ Ab_2 = e_2 \ \ldots \ Ab_n = e_n$

Finding A^{-1} is like solving in linear systems.

Find b_i such that $Ab_i = e_i$ for $i = 1, 2, \ldots n$

From (D), $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & 1 & -3 \end{array}\right]$

1st column

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 3 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} x = 1 \\ y = 0 \\ z = 0 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1$$

2nd column

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{array}{l} x = -1 \\ y = -1 \\ z = 1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad A\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = e_2$$

$$A\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = e_3$$

According to $(*)$, $A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 4 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix} = I_3$

Inverse Property, with Transpose

★ If A, B are both invertible $n \times n$ matrices

Then both AB and A^T are invertible

1. $(AB)^{-1} = B^{-1}A^{-1}$
2. $(A^T)^{-1} = (A^{-1})^T$

Pf of 1.

$$
\begin{aligned}
(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\
&= B^{-1} I_n B \\
&= B^{-1}B \\
&= I_n
\end{aligned}
$$

$\Rightarrow AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Inverse

8.1) The matrix $\begin{bmatrix} 6 & 1 \\ -8 & k \end{bmatrix}$ is invertible if and only if

ref has no zero entries in diagonal

$-8+(4/3)6=0 \qquad k+(4/3)=0$ when $k=-4/3$

ref has no zero entries, and thus matrix is invertible when $k \neq -4/3$

8.7) Are the following matrices invertible?

a) $\begin{bmatrix} 8 & 7 \\ -2 & 9 \\ 4 & 7 \end{bmatrix}$ No

b) $\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ -8 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$ Yes

c) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ -7 & -7 & -6 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -7 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ Yes

d) $\begin{bmatrix} -3 & 5 & -2 \\ 0 & 0 & 0 \\ 7 & 2 & -9 \end{bmatrix}$ No. makes det = 0

e) $\begin{bmatrix} 4 & -6 & 3 \\ -3 & 4 & -2 \\ 5 & -8 & 5 \end{bmatrix}$ Yes

f) $\begin{bmatrix} -1 & -9 & 8 \\ -5 & -2 & -3 \\ 0 & 4 & -4 \end{bmatrix}$ No

~ ref using MatLab

8.10) Which of the following is True for all $n \times n$ invertible matrices A and B?

a) $(A+B)(A-B) = A^2 - B^2$ False. $= A^2 - AB + BA - B^2$

b) $A + B$ is invertible False. counter e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c) A^7B^6 is invertible True.

d) $ABA^{-1} = B$ False. $ABA^{-1} \neq (BAA^{-1} = BI_n)$

e) $(I_n - A)(I_n + A) = I_n - A^2$ True. $I_n^2 = I_n, \ +AI_n - AI_n \ - A^2$

8.11) f) The matrix B^T is invertible True. B^{-1} exists. $(B^{-1})^T = (B^T)^{-1}$ exists

g) The columns of A form a basis for $\mathbb{R}^n$. True, lin independent

h) A can have 2 identical rows. False. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 9 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 9 & 6 \end{bmatrix}$, no ref has non-zero diag.

i) $AB = I$ does not imply $BA = I$. False. $AB = I$ means $A^{-1} = B$ and $B^{-1} = A$.

j) The system $B\vec{v} = \vec{b}$ has a unique sol'n for every vector $\vec{b}$ in $\mathbb{R}^n$. True. See (g)

k) The matrix A^T can be reduced to the $n \times n$ identity matrix. True. See (g).

8.2) Find the inverse of AB if $A^{-1} = \begin{bmatrix} 5 & 5 \\ -5 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$.

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -10 & 11 \\ 5 & -2 \end{bmatrix}$$

8.3) a) Find A^{-1} if $A = \begin{bmatrix} -1 & 4 & 8 \\ 0 & -9 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -1 & 4 & 8 & 1 & 0 & 0 \\ 0 & -9 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & -8 & -1 & 0 & 0 \\ 0 & 1 & -2/9 & 0 & -1/9 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\begin{array}{l} ①+4② \\ ②+\frac{2}{9}③ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -80/9 & -1 & -4/9 & 0 \\ 0 & 1 & 0 & 0 & -1/9 & -2/9 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$①+\tfrac{80}{9}③ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -4/9 & -80/9 \\ 0 & 1 & 0 & 0 & -1/9 & -2/9 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & -4/9 & -80/9 \\ 0 & -1/9 & -2/9 \\ 0 & 0 & -1 \end{bmatrix}$$

b) Find B^{-1} if $B = \begin{bmatrix} -1 & 0 & 0 \\ 4 & -9 & 0 \\ 8 & 2 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 4 & -9 & 0 & 0 & 1 & 0 \\ 8 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ ②-4① \\ ③-8① \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -9 & 0 & 4 & 1 & 0 \\ 0 & 2 & -1 & 8 & 0 & 1 \end{array}\right]$$

$$③+\tfrac{2}{9}② \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -9 & 0 & 4 & 1 & 0 \\ 0 & 0 & -1 & 80/9 & 2/9 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -4/9 & -1/9 & 0 \\ 0 & 0 & 1 & -80/9 & -2/9 & -1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -4/9 & -1/9 & 0 \\ -80/9 & -2/9 & -1 \end{bmatrix}$$

8.8) Find C^{-1} if $C = \begin{bmatrix} 3 & -2 & -3 \\ 1 & -2 & 2 \\ 3 & -4 & 1 \end{bmatrix}$

$$C^{-1} = \begin{bmatrix} 3 & 7 & -5 \\ 2.5 & 6 & -4.5 \\ 1 & 3 & -2 \end{bmatrix}$$

8.9) Find D^{-1} if $D = \begin{bmatrix} -1 & 0 & 2 & -4 \\ 0 & -1 & 3 & 4 \\ 0 & -1 & 6 & 8 \\ 0 & -1 & 6 & 9 \end{bmatrix}$

$$D^{-1} = \begin{bmatrix} -1 & -2/3 & 22/3 & -20/3 \\ 0 & -2 & 1 & 6 \\ 0 & -1/3 & 5/3 & -4/3 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

both via Matlab

8.6) Find a formula for T^{-1} if $T(x_1, x_2) = (4x_1 + 3x_2, -5x_1 - 5x_2)$

$$T\vec{x} = \begin{bmatrix} 4x_1+3x_2 \\ -5x_1-5x^2 \end{bmatrix}, \quad T = \begin{bmatrix} 4 & 3 \\ -5 & -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ -5 & -5 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 0 & -5/4 & 5/4 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 0 & 1 & -1 & -4/5 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 4 & 0 & 4 & 12/5 \\ 0 & 1 & -1 & -4/5 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 3/5 \\ 0 & 1 & -1 & 4/5 \end{array}\right]$$

$$T^{-1} = \begin{bmatrix} 1 & 3/5 \\ -1 & 4/5 \end{bmatrix}$$

Hilroy

8.4) Consider the following 2 systems:

a) $\begin{cases} -4x - 2y = 2 \\ 2x - 3y = -3 \end{cases}$ b) $\begin{cases} -4x - 2y = 3 \\ 2x - 3y = -2 \end{cases}$

Find the inverse of the (common) coefficient matrix A of the two systems.

$$\left[\begin{array}{cc|cc} -4 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 0 & -8 & 1 & 2 \\ 2 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 0 & 1 & -1/8 & -1/4 \\ 1 & -3/2 & 0 & 1/2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -3/16 & 1/8 \\ 0 & 1 & -1/8 & -1/4 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -3/16 & 1/8 \\ -1/8 & -1/4 \end{bmatrix}$$

Find the solutions to the two systems by using the inverse.

ie Evaluate $A^{-1}\vec{b}$ where $\vec{b}$ represents RHS. For (a), $\vec{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$; for (b) $\vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

a) $\begin{bmatrix} -3/16 & 1/8 \\ -1/8 & -1/4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/2 \end{bmatrix}$ b) $\begin{bmatrix} -3/16 & 1/8 \\ -1/8 & -1/4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -13/16 \\ 1/8 \end{bmatrix}$

Matlab

8.5) Find X if $\begin{bmatrix} 9 & -2 \\ 7 & -1 \end{bmatrix} X + \begin{bmatrix} -6 & -6 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} -9 & -7 \\ -3 & 6 \end{bmatrix}$

$$\begin{bmatrix} 9 & -2 \\ 7 & -1 \end{bmatrix} X = \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix}$$

$$A \quad X = b$$

$$A^{-1} = \begin{bmatrix} -1/5 & 2/5 \\ -7/5 & 9/5 \end{bmatrix} \text{ by MatLab}$$

$$A^{-1}b = \begin{bmatrix} 1/5 & -3/5 \\ 12/5 & -11/5 \end{bmatrix} = X$$

Inverse: geometric

Let $L: 3y = 4x$ be a line in $\mathbb{R}^2$. Then:

$\vec{u} = \frac{1}{5}[3\ 4]$

$A = \text{proj}_L = \begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$ $\quad B = \text{ref}_L = \begin{bmatrix} 2a_1^2 - 1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2 - 1 \end{bmatrix} = \begin{bmatrix} -7/25 & 24/25 \\ 24/25 & 7/25 \end{bmatrix}$

Are they invertible? If so, find their inverses.

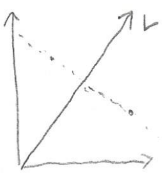
Method 1: $[A \mid I_n] \to [\text{rref } A \mid A^{-1}]$

$[B \mid I_n] \to [\text{rref } B \mid B^{-1}]$

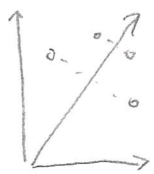
Note: $\det(A) = 0 \Rightarrow A$ is not invertible

$\det(B) = 1 \Rightarrow B$ is invertible

Method 2: Geometrically,



proj_L is not a reversible process.
A is not an invertible matrix.



ref_L is a reversible process. $\quad \text{Ref}_L \circ \text{Ref}_L = \text{Identity}$

$B \cdot B = I_2$,

$B^{-1} = B$

Similarly, $\text{Rot}_\phi^{-1} = \text{Rot}_{(-\phi)}$

Inverse: formula for 2x2 matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) \neq 0$, Then

$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

e.g. $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad \det(A) = -3$

$A^{-1} = \frac{1}{-3} \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 \\ 1 & 0 \end{bmatrix}$

Check. $\quad \begin{bmatrix} -2/3 & 1/3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$A^{-1}A = I_n$

$AA^{-1} = I_n$ $\quad \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2/3 & 1/3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Rmk: There are formulas for inverses of bigger matrixs. (Add'nl Topic 4.8.13)

If $\det(A) = 0$, A is not invertible.