Random Walks & Probability

① Weather

- if it rains on a given day, chance of rain next day = ½
- if it doesn't rain on a given day, chance of rain next day = ¼.

Two States: 1 - Rains 2 - Doesn't Rain / Dry

Let p_{ij} = probability of going from state $j \to i$

$p_{11} = \frac{1}{2}$ $p_{12} = \frac{1}{4}$

$p_{21} = 1 - \frac{1}{2} = \frac{1}{2}$ $p_{22} = 1 - \frac{1}{4} = 3/4$

Let $x_{n,1}$ = the probability that it rains on the n'th day

$x_{n,2}$ = the probability that it doesn't rain on the n'th day

$\vec{x_n} = \begin{bmatrix} x_{n,1} \\ x_{n,2} \end{bmatrix}$ = probability vector of weather on n'th day.
sum of entries = 1

$$\underbrace{x_{n+1,1}}_{\text{(prob) rains on (n+1)'th day}} = \underbrace{p_{11} x_{n,1}}_{\text{(prob) rains on both n and n+1}} + \underbrace{p_{12} x_{n,2}}_{\text{(prob) rains on n+1 after dry n day}}$$

$$\underbrace{x_{n+1,2}}_{\text{(prob) dry on (n+1)'th day}} = \underbrace{p_{21} x_{n,1}}_{\text{(prob) dry on n+1 after rain on n}} + \underbrace{p_{22} x_{n,2}}_{\text{(prob) dry on both n and n+1}}$$

$$\vec{x_{n+1}} = P\vec{x_n} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_{n,1} \\ x_{n,2} \end{bmatrix} = \begin{bmatrix} p_{11} x_{n,1} + p_{12} x_{n,2} \\ p_{21} x_{n,1} + p_{22} x_{n,2} \end{bmatrix} = \begin{bmatrix} x_{n+1,1} \\ x_{n+2,2} \end{bmatrix}$$

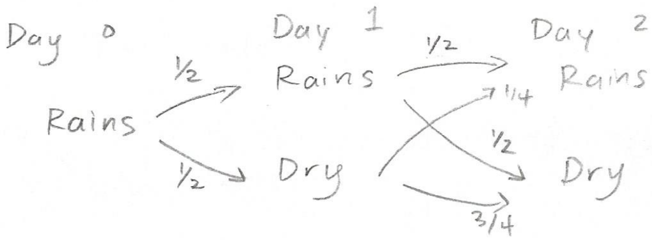
- Suppose that it rains today (0'th day). Find (prob) rain today and tmr.

$\vec{x_0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 3/4 \end{bmatrix}$ $\vec{x_{n+1}} = P\vec{x_n}$

$\vec{x_1} = P\vec{x_0} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\vec{x_2} = P\vec{x_1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$ $(1/2 \times \frac{1}{2}) + (1/4 \times 1/2)$

$\vec{x_n} = P\vec{x_{n-1}}$
$= P(P\vec{x_{n-2}}) = P^2 \vec{x_{n-2}}$
$= P(P(P\vec{x_{n-3}})) = P^3 \vec{x_{n-3}}$
$= P^n \vec{x_{n-n}} = P^n \vec{x_0}$



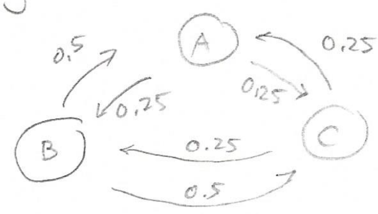
- $\vec{x_n} = P^n \vec{x_0}$ In our example, given $P^{100} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}^{100} \approx \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$ ...equal columns

If $\vec{x_0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\vec{x_{100}} = P^{100} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

If $\vec{x_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\vec{x_{100}} = P^{100} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

Weather 100 days later is almost independent of today's weather.

② Migration : 3 islands (states), people move b/w them



Arrows represent portion of ppl moving from one island to another each year.

from state/island

$$P = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} A & B & C \end{array}$$

$$P = \left[\begin{array}{ccc} \boxed{0.5} & 0.5 & 0.25 \\ 0.25 & \boxed{0} & 0.25 \\ 0.25 & 0.5 & \boxed{0.5} \end{array}\right] \begin{array}{c} A \\ B \\ C \end{array}$$

to state/island

$\boxed{}$ calculated

sum of column = 1

- Suppose initial population is A: 800, B: 800, C: 400.
  Predict population for the following 2 years.

Let $\vec{x_n} = \begin{bmatrix} x_{n,1} \\ x_{n,2} \\ x_{n,3} \end{bmatrix}$ denote pop on A, B, C resp. in the n'th year $\qquad \vec{x_0} = \begin{bmatrix} 800 \\ 800 \\ 400 \end{bmatrix}$

$$x_{n+1,1} = 0.5\,x_{n,1} + 0.5\,x_{n,2} + 0.25\,x_{n,3}$$

$$\vec{x_{n+1}} = P\vec{x_n}$$

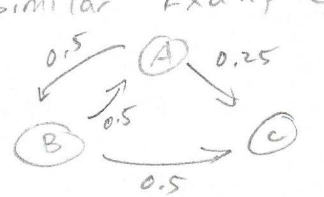
$$\vec{x_1} = P\vec{x_0} = \begin{bmatrix} 0.5 & 0.5 & 0.25 \\ 0.25 & 0 & 0.25 \\ 0.25 & .5 & 0.5 \end{bmatrix}\begin{bmatrix} 800 \\ 800 \\ 400 \end{bmatrix} = \begin{bmatrix} 900 \\ 300 \\ 800 \end{bmatrix}$$

$$\vec{x_2} = \vec{P_2}\,\vec{x_0} = P\vec{x_1} = \frac{1}{16}\begin{bmatrix} 7 & 6 & \boxed{6} \\ 3 & 4 & 3 \\ 6 & 6 & 7 \end{bmatrix}\begin{bmatrix} 800 \\ 800 \\ 400 \end{bmatrix} = \begin{bmatrix} 800 \\ 425 \\ 775 \end{bmatrix}$$

Rmk: $6/16$ is probability ppl on island C will move to island A in 2 years.

- $P^{100} \simeq \begin{bmatrix} 0.4 & 0.4 & 0.4 \\ 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$ Population 100 years later is almost independent of initial ratios. Same columns

- Similar Example



$$\vec{x_0} = \begin{bmatrix} 800 \\ 800 \\ 400 \end{bmatrix} \qquad P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0 & 0 \\ 0.25 & 0.5 & 1 \end{bmatrix} \qquad P^{100} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\vec{x_{100}} = P^{100}\vec{x_0} = \begin{bmatrix} 0 \\ 0 \\ 2000 \end{bmatrix}$$

Transition Matrix

n-state problem ⇒ transition matrix is square $n \times n$ matrix.

$$P = \begin{array}{c} \\ \end{array} \begin{array}{cccc} & & \boxed{\text{From}} & \\ 1 & 2 & \cdots & n \end{array}$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} \ \boxed{\text{To}} \qquad \vec{x_{k+1}} = P\vec{x_k}$$

p_{ij} : (prob) $j \to i$

p_{ij} on P^k = (prob) starting at j and ending at i after k steps

① Sum of each column = 1

② each entry of $P \geq 0$

e.g. $6/16$ of people on C will be on A after 2 years

Random Walks

7.10 Each decade, 50% of pop. in city moves to surrounding suburbs and 10% of ppl in suburbs move to the city.

Currently, pop. city = 7 million, total = 16 million.

pop. suburbs = 9 million

$$\vec{x}_0 = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \qquad P = \begin{array}{cc} c & s \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} & \begin{array}{c} c \\ s \end{array} \end{array}$$

Compute the expected pop. after 3 decades.

$$P^3 = \begin{bmatrix} 0.22 & 0.156 \\ 0.78 & 0.844 \end{bmatrix} \text{ by Matlab/Octave}$$

$$P^3 \vec{x}_0 = \begin{bmatrix} 0.22 & 0.156 \\ 0.78 & 0.844 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 11.6 \end{bmatrix} \text{ by Matlab/Octave}$$

7.11 Consider a random walk with 2 states, where prob staying in location 1 is 0.88, and once the walker has reached location 2 he will never go back to 1

Find the transition matrix P.

$$P = \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 0.88 & 0 \\ 0.12 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Find P^n for arbitrary positive integer n.

$$\begin{bmatrix} 0.88 & 0 \\ 0.12 & 1 \end{bmatrix} \begin{bmatrix} 0.88 & 0 \\ 0.12 & 1 \end{bmatrix} = \begin{bmatrix} 0.7744 & 0 \\ 0.2256 & 1 \end{bmatrix} \quad \text{by Matlab}$$

$$[\quad " \quad] \begin{bmatrix} 0.88 & 0 \\ 0.12 & 1 \end{bmatrix} = \begin{bmatrix} 0.68147 & 0 \\ 0.31853 & 1 \end{bmatrix}$$

$$P^n = \begin{bmatrix} 0.88^n & 0 \\ 1-0.88^n & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.88 & 0 \\ 0.12 & 1 \end{bmatrix} \begin{bmatrix} 0.88 & 0 \\ 0.12 & 1 \end{bmatrix} = \begin{bmatrix} 0.88^2+0 & 0 \\ 0.88\times0.12+0.12 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.88^2 & 0 \\ 0.88\times0.12+0.12 & 1 \end{bmatrix} \begin{bmatrix} 0.88 & 0 \\ 0.12 & 1 \end{bmatrix} = \begin{bmatrix} 0.88^3 & 0 \\ 0.88^2\times0.12+0.12(0.88) & 1 \end{bmatrix}$$

Meh. Columns add to 1.

7.12 A group insurance plan allows 3 diff options for participant A, B or C. Suppose at present the %s of total # of participants enrolled in each plan are 30, 20, 50 resp. Using info from a survey, assume participants will change plans annually as shown:

$\vec{X_0} = \begin{bmatrix} .3 \\ .2 \\ .5 \end{bmatrix}$

%	from A	B	C
to A	70	60	30
B	10	30	60
C	20	10	10

$P = \begin{bmatrix} 0.7 & 0.6 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.2 & 0.1 & 0.1 \end{bmatrix}$

Find the % participants enrolled in each plan after 2 years.

$P^2 = \begin{bmatrix} 0.61 & 0.63 & 0.6 \\ 0.22 & 0.21 & 0.27 \\ 0.17 & 0.16 & 0.13 \end{bmatrix}$ by Matlab

$P^2 \vec{X_0} = \begin{bmatrix} 0.609 \\ 0.243 \\ 0.148 \end{bmatrix}$ 60.9 % / 24.3 % / 14.8 %

7.13 Infection originated from unknown pollution can kill 70% of those already ill. Makes 30% of healthy pop. ill each month. Assume everyone initially healthy. Assume death rate of healthy pop negligible in months.

a) Suppose no one who is infected can be healthy again. Find % pop who die in next 4 months.

healthy
ill
dead

$$P = \begin{array}{c} \begin{array}{ccc} H & I & D \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0 \\ 0.3 & 0.3 & 0 \\ 0 & 0.7 & 1 \end{bmatrix} \end{array} \begin{array}{c} H \\ I \\ D \end{array} \qquad \vec{X_0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad P^4 \vec{X_0} = \begin{bmatrix} 0.2401 \\ 0.174 \\ 0.5859 \end{bmatrix}$$

58.6% dead by Matlab

b) Cure found, cures 20% of infected, reduces death of infected to 20%. Find % who die 4 mo.

$$P = \begin{bmatrix} 0.7 & 0.2 & 0 \\ 0.3 & 0.6 & 0 \\ 0 & 0.2 & 1 \end{bmatrix} \qquad \vec{X_0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad P^4 \vec{X_0} = \begin{bmatrix} 0.4039 \\ 0.3783 \\ 0.2178 \end{bmatrix}$$

21.78% dead

c) Suppose those cured in (b) indefinitely immune. Find % die in next 4 months. Note you will need to introduce a new subgroup for immune population.

healthy
~~silck~~ sick
immune
dead

$$P = \begin{array}{c} \begin{array}{cccc} H & S & I & D \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0.2 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} H \\ S \\ I \\ D \end{array} \qquad \vec{X_0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad P^4 \vec{X_0} = \begin{bmatrix} 0.2401 \\ 0.3315 \\ 0.2142 \\ 0.2142 \end{bmatrix}$$

21.42% dead.