Linear Transformations

$f:\mathbb{R}\to\mathbb{R}$
$f(x)=3x+2$
$f(1)=5$

Consider functions. Input a #, get a #.

A **transformation** is another sort of machinery, which input & output is vectors.

$T:\mathbb{R}^n \to \mathbb{R}^m$ e.g. $T:\mathbb{R}^2\to\mathbb{R}^3$
$T(x,y)=(x^2+1,\ x+y,\ y)$
$T([1,1])=([2,2,1])$

T well behaved in vec add & scalar ×

★ A transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ is **linear** if for any 2 input vectors x and y

$$T(s\vec{x}+t\vec{y})=sT(\vec{x})+tT(\vec{y}) \Rightarrow \begin{array}{l} ① \ T(\vec{x}+\vec{y})=T(\vec{x})+T(\vec{y}) \\ ② \ T(s\vec{x})=sT(\vec{x}) \end{array}$$

e.g. Show that a) $T(x_1,x_2)=(x_1+1,\ x_2+1)$ and b) $T(x_1,x_2)=x_1^2$ are not linear

a) $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=(2,2)$ $\quad T\left(\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}\right)=T\left(\begin{bmatrix}3\\3\end{bmatrix}\right)=(4,4)$
$T\left(\begin{bmatrix}2\\2\end{bmatrix}\right)=(3,3)$

$(4,4)\neq(2,2)+(3,3)$

T does not satisfy $T(\vec{x}+\vec{y})=T(\vec{x})+T(\vec{y}) \Rightarrow$ it is not linear

b) $T\left(10\begin{bmatrix}1\\1\end{bmatrix}\right)=T\left(\begin{bmatrix}10\\10\end{bmatrix}\right)=100$

$10\,T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=10$

T does not satisfy $T(s\vec{x})=sT(\vec{x}) \Rightarrow$ it is not linear.

★ Let A be an $m\times n$ matrix.
$T:\mathbb{R}^n\to\mathbb{R}^m$ be given by $T(\vec{x})=A\vec{x}$ $\quad (m\times n)(n\times 1)$ $\quad m\times 1$ output

Show that T is linear, from the definition.

① $T(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A\vec{x}+A\vec{y}$
$T(\vec{x}+\vec{y}) \qquad = T(\vec{x})+T(\vec{y})$

② $T(s\vec{x}) = A(s\vec{x}) = s(A\vec{x})$
$T(s\vec{x}) \qquad = sT(\vec{x})$

(1 & 2) $T(s\vec{x}+t\vec{y})=A(s\vec{x}+t\vec{y})$
$=A(s\vec{x})+A(t\vec{y})$
$=s(A\vec{x})+t(A\vec{y})$

This gives us ③ Linear transformations can be expressed as a matrix (multiplication).

Suppose T is a lin transformation $T:\mathbb{R}^n\to\mathbb{R}^m$

★★ Let A be the $m\times n$ matrix given by

$$A=\begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \quad e_1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} \quad e_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \quad e_n=\begin{bmatrix}0\\0\\\vdots\\0\\n\end{bmatrix}$$

Then $T(\vec{x})=A\vec{x}$

Why? In general,

$$\underset{m\times n}{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}\ \underset{n\times 1}{\begin{bmatrix}x_1\\x_2\\|\\x_n\end{bmatrix}} = x_1\underset{m\times 1}{\begin{bmatrix}|\\v_1\\|\end{bmatrix}}+x_2\underset{m\times 1}{\begin{bmatrix}|\\v_2\\|\end{bmatrix}}+\cdots+x_n\underset{m\times 1}{\begin{bmatrix}|\\v_n\\|\end{bmatrix}} = \underset{m\times 1}{x_1v_1+x_2v_2+\cdots+x_nv_n}$$

See: $\vec{x}=\begin{bmatrix}x_1\\x_2\\x_n\end{bmatrix}=\begin{bmatrix}x_1\\0\\\vdots\\0\end{bmatrix}+\begin{bmatrix}0\\x_2\\\vdots\\0\end{bmatrix}+\cdots+\begin{bmatrix}0\\0\\\vdots\\x_n\end{bmatrix}=x_1\vec{e}_1+x_2\vec{e}_2+\cdots+x_n\vec{e}_n$

observe (e.g.): $\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}7(1)+8(2)+9\\7(4)+8(5)+9(\end{bmatrix}$
$=7\begin{bmatrix}1\\4\end{bmatrix}+8\begin{bmatrix}2\\5\end{bmatrix}+9\begin{bmatrix}3\\6\end{bmatrix}$

$T(\vec{x})=T(x_1\vec{e}_1+x_2\vec{e}_2+\cdots+x_n\vec{e}_n)$
$=x_1T(\vec{e}_1)+x_2T(\vec{e}_2)+\cdots+x_nT(\vec{e}_n)$ T is linear
$=\begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_n\end{bmatrix}$

linear trans $\mathbb{R}^n\to\mathbb{R}^m$ $\Leftrightarrow$ $m\times n$ matri e.g. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a lin T given by $T(x_1, x_2, x_3) = (2x_1 + x_3, x_2 - x_3)$

Find the matrix associated to T.

$\mathbb{R}^3 \to \mathbb{R}^2$, the matrix is 2×3, given by

$$A = \begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix}$$

$T(e_1) = T(1,0,0) = (2,0)$

$T(e_2) = (0,1)$

$T(e_3) = (1,-1)$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

e.g. $T(1,1,1) = (3,0)$

$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$

$T(x_1, x_2, x_3) = (2x_1 + x_3, x_2 - x_3)$

$A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2x_1 + x_3\\x_2 - x_3\end{bmatrix}$

or coefficients! $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

e.g. Let T be a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ and

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}4\\7\end{bmatrix}$$

Find the matrix representing T and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

$\mathbb{R}^2 \to \mathbb{R}^2$, 2×2 matrix. Need $T(e_1)$ and $T(e_2)$.

$$T(e_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$$

$$e_2 = \begin{bmatrix}0\\1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix}$$ express e_2 as a lin combo

$$\left.\begin{aligned} T(e_2) &= T\left(\tfrac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}\right) \\ &= \tfrac{1}{2}T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - \tfrac{1}{2}T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \\ &= \tfrac{1}{2}\begin{bmatrix}4\\7\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}2\\1\end{bmatrix} \\ &= \begin{bmatrix}1\\3\end{bmatrix} \end{aligned}\right\}$$ We can do all this because we know T is linear.

The matrix representing $T = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$$

Try to contradict def'n of linear transformation

(7.1) Are these linear transformations?

a) $T(x_1, x_2) = (2x_1 - 3x_2,\ x_1 + 4,\ 5x_2)$

$T(2\begin{bmatrix}3\\1\end{bmatrix}) = T(\begin{bmatrix}6\\2\end{bmatrix}) = (6, 10, 10)$

$2T(\begin{bmatrix}3\\1\end{bmatrix}) = 2(3, 7, 5) = (6, 14, 10)$ No.

b) $T(x_1, x_2) = (4x_1 - 2x_2,\ 3|x_2|)$

① $T(2\begin{bmatrix}2\\1\end{bmatrix}) = T(\begin{bmatrix}4\\2\end{bmatrix}) = (12, 6)$

$2T(\begin{bmatrix}2\\1\end{bmatrix}) = 2(6, 3) = (12, 6)$

② $T(\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}1\\7\end{bmatrix}) = T(\begin{bmatrix}3\\8\end{bmatrix}) = (-4, 24)$

$T(\begin{bmatrix}2\\1\end{bmatrix}) + T(\begin{bmatrix}1\\7\end{bmatrix}) = (6, 3) + (-10, 21) = (-4, 24)$

③ $T(2\begin{bmatrix}1\\-2\end{bmatrix}) = T(\begin{bmatrix}2\\-4\end{bmatrix}) = (16, 12)$

$2T(\begin{bmatrix}1\\-2\end{bmatrix}) = 2(8, 6) = (16, 12)$

④ $T(\begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}2\\-3\end{bmatrix}) = T(\begin{bmatrix}3\\-5\end{bmatrix}) = (22, 15)$

$T(\begin{bmatrix}1\\-2\end{bmatrix}) + T(\begin{bmatrix}2\\-3\end{bmatrix}) = \begin{bmatrix}8\\6\end{bmatrix} + \begin{bmatrix}14\\9\end{bmatrix} = \begin{bmatrix}22\\15\end{bmatrix}$

⑤ $T(\begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}) = T(\begin{bmatrix}2\\0\end{bmatrix}) = (8, 0)$

$T(\begin{bmatrix}1\\-2\end{bmatrix}) + T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}8\\6\end{bmatrix} + \begin{bmatrix}0\\6\end{bmatrix} = \begin{bmatrix}8\\12\end{bmatrix}$

No!

c) $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$

$T([1\ 1\ 1\ 1]) = 5$

$T([2\ 2\ 2\ 2]) = T(2[1\ 1\ 1\ 1]) = 10 = 2T([1\ 1\ 1\ 1])$

$T([1\ 1\ 1\ 1] + [2\ 1\ 2\ 3]) = T([3\ 2\ 3\ 4]) = 13 = 5 + T([2\ 1\ 2\ 3])$ Yes.

d) $T(x_1, x_2, x_3) = (1,\ x_2,\ x_3)$ No!

$T([1\ 1\ 1]) = (1, 1, 1)$

$T([2\ 2\ 2]) = (1, 2, 2)$ $T(2[1\ 1\ 1]) \neq 2T([1, 1, 1])$

e) $T(x_1, x_2, x_3) = (x_1,\ 0,\ x_3)$

$T([1, 1, 1]) = (1, 0, 1)$

$T([2, 2, 2]) = (2, 0, 2)$ $T(2[1, 1, 1]) = 2T([1, 1, 1])$

$T([-1\ 1\ 2] + [2\ 1\ 0]) = T([1, 2, 2]) = (1, 0, 2)$

$T([-1, 1, 2]) + T([2, 1, 0]) = (-1, 0, 2) + (2, 0, 0) = (1, 0, 2)$ Yes.

f) $T(x_1, x_2, x_3) = (x_1,\ x_2 - x_3)$

$T([1\ 2\ 3]) = (1, -1)$ $T([1\ 0\ 2] + [2\ 1\ 0]) = T([3\ 1\ 2]) = (3, -1)$ Yes

$T([2\ 4\ 6]) = (2, -2)$ ✓ $T([1\ 0\ 2]) + T([2\ 1\ 0]) = (1, -2) + (2, 1) = (3, -1)$

(7.4) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the lin trans. which ~~define~~ satisfies

$$T\left(\begin{bmatrix}-5\\-1\end{bmatrix}\right) = \begin{bmatrix}0\\-3\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}-4\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$$

a) Find $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

$$\begin{bmatrix}1\\0\end{bmatrix} = s\begin{bmatrix}-5\\-1\end{bmatrix} + t\begin{bmatrix}-4\\-1\end{bmatrix}$$

$$= 1\begin{bmatrix}-4\\-1\end{bmatrix} - 1\begin{bmatrix}-5\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 1T\begin{pmatrix}-4\\-1\end{pmatrix} - 1T\begin{pmatrix}-5\\-1\end{pmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\-3\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

b) Find $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

$$\begin{bmatrix}0\\1\end{bmatrix} = 4\begin{bmatrix}-5\\-1\end{bmatrix} - 5\begin{bmatrix}-4\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 4T\begin{pmatrix}-5\\-1\end{pmatrix} - 5T\begin{pmatrix}-4\\-1\end{pmatrix} = 4\begin{bmatrix}0\\-3\end{bmatrix} - 5\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-5\\-17\end{bmatrix}$$

c) Find the matrix representation of T

$$\begin{bmatrix}| & |\\ T(e_1) & T(e_2)\\ | & |\end{bmatrix} = \begin{bmatrix}1 & -5\\4 & -17\end{bmatrix}$$

d) Find $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$ for arbitrary x and y

$$\begin{bmatrix}1 & -5\\4 & -17\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\4\end{bmatrix} + y\begin{bmatrix}-5\\-17\end{bmatrix} = \begin{bmatrix}x-5y\\4x-17y\end{bmatrix}$$

(7.8) Let $A = \begin{bmatrix}2 & 8\\4 & 16\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}-24\\k\end{bmatrix}$

Find k so there exists a vector $\vec{x}$ whose image under the lin trans $T(\vec{x}) = A\vec{x}$ is $\vec{w}$.

$$x_1\begin{bmatrix}2\\4\end{bmatrix} + x_2\begin{bmatrix}8\\16\end{bmatrix} = \begin{bmatrix}-24\\k\end{bmatrix} \qquad x_1 = -12 - 4x_2$$

$$-48 - 16x_2 + 16x_2 = k \qquad k = -48$$

Find k so that $\vec{w}$ is a solution of the equation. $A\vec{x} = \vec{0}$.

$$\begin{bmatrix}2 & 8\\4 & 16\end{bmatrix}\begin{bmatrix}-24\\k\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$-48 + 8k = 0 \qquad k = 48/8 = 6$$

$$-96 + 16k = 0 \qquad k = 96/16 = 6$$

$$A = \begin{bmatrix} 2 & 8 \\ 4 & 16 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} -24 \\ k \end{bmatrix}$$

Find k so there exists a vector $\vec{x}$ whose imge under the lin. trans. $T(\vec{x}) = A\vec{x}$ is $\vec{w}$

$$\begin{bmatrix} 2 & 8 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ k \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 8 & -24 \\ 4 & 16 & k \end{array}\right]$$

$2x_1 + 8x_2 = -24$

$4x_1 + 16x_2 = k$

$(-48 - 16x_2) + 16x_2 = k$, $k = -48$

Find k so that $\vec{w}$ is a sol'n of the eq'n $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{cc|c} 2 & 8 & 0 \\ 4 & 16 & 0 \end{array}\right]$$

$$\begin{bmatrix} 2 & 8 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} -24 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2(-24) + 8k = 0$ $\qquad k = 6$

$4(-24) + 16k = 0$

(7.2) Find the matrix of the following linear transformations:

a) $Q: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad Q(\vec{u}) = \begin{bmatrix} 8u_1 - 4u_2 \\ -6u_1 + u_2 \end{bmatrix}$ $\quad \begin{bmatrix} 8 & -4 \\ -6 & 1 \end{bmatrix}$

b) $R: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad R(\vec{v}) = \begin{bmatrix} 4v_1 - v_2 + 2v_3 \\ -8v_2 - 4v_3 \end{bmatrix}$ $\quad \begin{bmatrix} 4 & -1 & 2 \\ 0 & -8 & -4 \end{bmatrix}$

c) $S: \mathbb{R}^4 \to \mathbb{R}^3$ $\quad S(\vec{w}) = \begin{bmatrix} 4w_1 + 7w_4 \\ 4w_1 - 5w_2 \\ -6w_4 \end{bmatrix}$ $\quad \begin{bmatrix} 4 & 0 & 0 & 7 \\ 4 & -5 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$

d) $T: \mathbb{R}^4 \to \mathbb{R}$ $\quad T(\vec{x}) = [5x_1 - 4x_2 + 2x_3]$ $\quad [5 \;\; -4 \;\; 2 \;\; 0]$

(7.3) Let $A = \begin{bmatrix} -2 & 5 & 3 \\ -5 & -2 & 6 \\ 4 & -3 & -4 \\ -2 & -3 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 26 \\ 14 \\ -21 \\ 17 \end{bmatrix}$.

Define the lin transform. $T: \mathbb{R}^3 \to \mathbb{R}^4$ as $T(\vec{x}) = A\vec{x}$.

Find a vector $\vec{x}$ whose image under T is $\vec{b}$

Note: image: what comes out of the transformation

$$x_1 \cdot \begin{bmatrix} -2 \\ -5 \\ 4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -2 \\ -3 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 26 \\ 14 \\ -21 \\ 17 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -2 & 5 & 3 & 26 \\ -5 & -2 & 6 & 14 \\ 4 & -3 & -4 & -21 \\ -2 & -3 & 6 & 17 \end{array}\right]$$

4 eqns.
3 unknowns
unlikely >1 soln

$$-2x_1 + 5x_2 + 3x_3 = 26$$
$$-5x_1 - 2x_2 + 6x_3 = 14$$
$$4x_1 - 3x_2 - 4x_3 = -21$$
$$-2x_1 - 3x_2 + 6x_3 = 17$$

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & -8 & 3 & -9 \\ -4 & 3 & 4 & 21 \\ -5 & -2 & 6 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & -8 & 3 & -9 \\ 0 & -7 & -2 & -31 \\ -10 & -4 & 12 & 28 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & -8 & 3 & -9 \\ 0 & -7 & -2 & -31 \\ 0 & -29 & -3 & -102 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & 29 & 3 & 102 \\ 0 & 56 & -21 & 63 \\ 0 & -7 & -2 & -31 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & 29 & 3 & 102 \\ 0 & -7 & -2 & -31 \\ 0 & 0 & -37 & -185 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & 203 & 21 & 714 \\ 0 & -7 & -2 & -31 \\ 0 & 0 & 37 & 185 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & 0 & -37 & -185 \\ 0 & 0 & 37 & 185 \\ 0 & 203 & 21 & 714 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -5 & -3 & -26 \\ 0 & 203 & 21 & 714 \\ 0 & 0 & 37 & 185 \end{array}\right]$$

unique soln

$37x_3 = 185$
$x_3 = 5$

$203x_2 + 21(5) = 714$
$x_2 = 609/203$
$= 3$

$2x_1 - 5(3) - 3(5) = -26$
$2x_1 = 4$
$x_1 = 2$

$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Hilroy

WW7.9 Let $A = \begin{bmatrix} 9 & -6 & 3 \\ 12 & -8 & 4 \end{bmatrix} = \left[\begin{array}{ccc|c} 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{array}\right]$

Find two linearly independent vectors $\vec{u}$ and $\vec{v}$ that solve $A\vec{x} = \vec{0}$

$\vec{u} = [1, 1, -1]$ $\vec{v} = [2, 3, 0]$

Write a non-zero vector $\vec{w}$ so that $A\vec{x} = \vec{w}$ for some vector $\vec{x}$.

$\vec{w} = [3, 4]$

$3a - 2b + 1 = c$

Common Useful Linear Transformations

a_1, a_2 components of $\hat{a}$

1. dilation, contraction ~~~ kI_n
2. projection ~~in $\mathbb{R}^2$~~ ~~~ $\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}$ $\quad (\vec{a}\cdot\vec{x})\vec{x}$
3. rotations in $\mathbb{R}^2$ ~~~ $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$
4. reflection in $\mathbb{R}^2$ ~~~ $\begin{bmatrix} 2a_1^2-1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2-1 \end{bmatrix}$ $\quad 2\mathrm{proj}_L\vec{x} - \vec{x}$

$= \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$

1. Dilation / Contraction

$T(\vec{x}) = k(\vec{x})$, k is a scalar

If $|k| > 1$, dilation
$\quad\quad < 1$, contraction

If $k < 0$, direction change

$T(\vec{x}) = (kI_n)(\vec{x})$

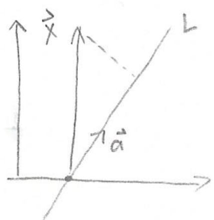
Matrix Form

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $T(\vec{x}) = k\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = kI_n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

2. Projection (in $\mathbb{R}^2$)

$\mathrm{proj}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \dfrac{\vec{a}\cdot\vec{x}}{\vec{a}\cdot\vec{a}}\vec{a}$



$= (\vec{a}\cdot\vec{x})\vec{a}$

$= (a_1x_1 + a_2x_2)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$= \begin{bmatrix} a_1^2x_1 + a_2a_1x_2 \\ a_1a_2x_1 + a_2^2x_2 \end{bmatrix}$

$= \begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Let $\|\vec{a}\| = 1$ (unit vector)
$\vec{a}\cdot\vec{a} = 1$
$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$\mathrm{proj}_L(\vec{x}) = \left(\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}\right)(\vec{x})$

$\mathbb{R}^3$:

$= \begin{bmatrix} a_1^2x_1 + a_1a_2x_2 + a_1a_3x_3 \\ a_2a_1x_1 + a_2^2x_2 + a_2a_3x_3 \\ a_3a_1x_1 + a_3a_2x_2 + a_3^2x_3 \end{bmatrix}$

$= \begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

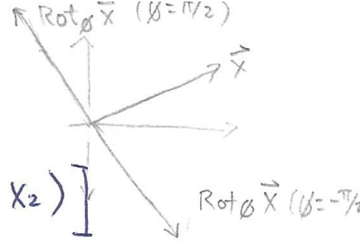
e.g. $L: 4y = 3x$ $\quad$ Find $\mathrm{proj}_L\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Find a vector on L: $4(3) = 3(4)$ $\quad \vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\quad \hat{v} = \sqrt{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

$\mathrm{proj}_L = \begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix} = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$

$\mathrm{proj}_L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 28/25 \\ 21/25 \end{bmatrix}$

③ Rotations in $\mathbb{R}^2$



$Rot_\phi : \mathbb{R}^2 \to \mathbb{R}^2$ rotates a vector about origin by ϕ ccw

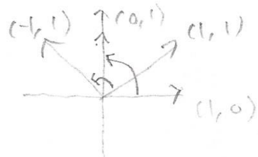
$$Rot_\phi(x_1, x_2) = [(\cos\phi\, x_1 - \sin\phi\, x_2),\ (\sin\phi\, x_1 + \cos\phi\, x_2)]$$

e.g. $\phi = \pi/2$, $\cos\phi = 0$, $\sin\phi = 1$

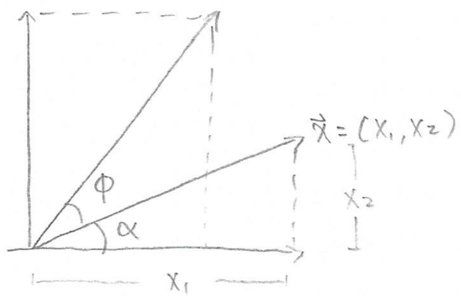
$Rot_{\pi/2}(x_1, x_2) = (-\sin\phi\, x_2,\ \sin\phi\, x_1)$

$Rot_{\pi/2}(x_1, x_2) = (-x^2, x_1)$

$Rot_{\pi/2}(1, 0) = (0, 1)$ $\qquad Rot_{\pi/2}(1,1) = (-1, 1)$



Derivation



$\vec{x} = (x_1, x_2)$

$= (\|\vec{x}\|\cos\alpha,\ \|\vec{x}\|\sin\alpha)$

$\|Rot_\phi \vec{x}\| = \|\vec{x}\|$

$\star$ $Rot_\phi \vec{x} = (\|\vec{x}\|\cos(\alpha+\phi),\ \|\vec{x}\|\sin(\alpha+\phi))$

Must relate x_1, x_2 to this

Recall: $\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$
$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \sin\phi\cos\alpha$

$$Rot_\phi \vec{x} = (\|x\|\cos(\alpha+\phi),\ \|x\|\sin(\alpha+\phi))$$

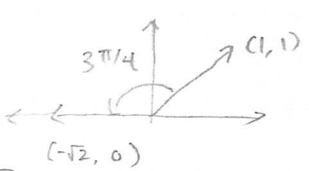
$$= (\underbrace{\|x\|\cos\alpha}_{x_1}\cos\phi - \underbrace{\|x\|\sin\alpha}_{x_2}\sin\phi,\ \underbrace{\|x\|\sin\alpha}_{x_2}\cos\phi + \underbrace{\|x\|}_{}\sin\phi\underbrace{\cos\alpha}_{x_1})$$

$$= (\cos\phi\, x_1 - \sin\phi\, x_2,\ \cos\phi\, x_2 + \sin\phi\, x_1)$$

$$\begin{bmatrix} \cos\phi\, x_1 - \sin\phi\, x_2 \\ \cos\phi\, x_2 + \sin\phi\, x_1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Rot_\phi(\vec{x}) = \left(\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\right)(\vec{x})$$

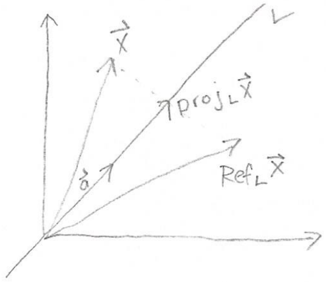
e.g $\phi = \frac{3\pi}{4}$ $\qquad Rot_\phi = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$



$$Rot_\phi \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$$

$$\left\|\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\|$$

④ Reflections



$\text{proj}_L \vec{x}$ is midpoint of $\vec{x}$ and $\text{Ref}_L \vec{x}$

$$\text{proj}_L \vec{x} = \tfrac{1}{2}\vec{x} + \tfrac{1}{2}\text{Ref}_L \vec{x}$$

$$\text{Ref}_L \vec{x} = 2\,\text{proj}_L \vec{x} - \vec{x}$$

In matrix form, $\text{Ref}_L \vec{x} = 2 \overbrace{\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\text{proj}_L \vec{x}} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= \begin{bmatrix} 2a_1^2 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2a_1^2 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2a_1^2-1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

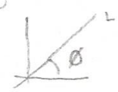
You could probably do it in 3D if you wanted to :)

$$\text{Ref}_L(\vec{x}) = \left(\begin{bmatrix} 2a_1^2-1 & 2a_1a_2 \\ 2a_1a_2 & 2a_2^2-1 \end{bmatrix}\right)(\vec{x})$$

Ref$_L$ special case

Show that Ref$_L$ is given by the matrix multiplication $\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$ using double-angle identities for the case:

Take $\vec{a} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$



- proj$_L$ given by $\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix} = \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}$

$$2\,\text{proj}_L = \begin{bmatrix} 2\cos^2\phi & 2\sin\phi\cos\phi \\ 2\sin\phi\cos\phi & 2\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} 1+\cos 2\phi & \sin 2\phi \\ \sin 2\phi & 1-\cos 2\phi \end{bmatrix}$$

$2\sin\phi\cos\phi = \sin(2\phi)$
$\cos^2\phi = (\tfrac{1}{2})(1+\cos(2\phi))$
$\sin^2\phi = (\tfrac{1}{2})(1-\cos(2\phi))$

- $\text{Ref}_L \vec{x} = 2\,\text{proj}_L \vec{x} - \vec{x} = \begin{bmatrix} 1+\cos 2\phi & \sin 2\phi \\ \sin 2\phi & 1-\cos 2\phi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- e.g. L given by $4y = 3x$. Find $\text{Ref}_L\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is a vector on L. $\vec{a} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$
  - $\text{Ref}_L \vec{x} = 2\,\text{proj}_L \vec{x} - \vec{x}$

$$2\,\text{proj}_L \vec{x} = 2\begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$$

$$= \begin{bmatrix} 32/25 & 24/25 \\ 24/25 & 18/25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{Ref}_L \vec{x} = \begin{bmatrix} 7/25 & 24/25 \\ 24/25 & -7/25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{Ref}_L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/25 & 24/25 \\ 24/25 & -7/25 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 31/25 \\ 17/25 \end{bmatrix}$$

6.12 Consider the line $x_2 = -7/4\, x_1$: L

a) Find the matrix A which projects a vector $\vec{x}$ onto the line L

vector on L: $\begin{bmatrix}4\\-7\end{bmatrix}$, unit vector $= \frac{1}{\sqrt{65}}\begin{bmatrix}4\\-7\end{bmatrix}$

$\text{proj}_L = \begin{bmatrix}a_1^2 & a_1a_2\\ a_1a_2 & a_2^2\end{bmatrix} = \begin{bmatrix}16/65 & -28/65\\ -28/65 & 49/65\end{bmatrix}$

b) Find the vector produced by projecting $\begin{bmatrix}4\\4\end{bmatrix}$ onto the line L

$\text{proj}_L \begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}16/65 & -28/65\\ -28/65 & 49/65\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}-48/65\\ 84/65\end{bmatrix}$

c) Find the matrix B which reflects a vector $\vec{x}$ about the line L

$\text{Ref}_L = 2\,\text{proj}_L - \begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}-33/65 & -56/65\\ -56/65 & 33/65\end{bmatrix} = \begin{bmatrix}2a_1^2-1 & 2a_1a_2\\ 2a_1a_2 & 2a_2^2-1\end{bmatrix}$

d) Find the vector produced by reflecting $\begin{bmatrix}4\\4\end{bmatrix}$ about the line L

$\text{Ref}_L \begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}-33/65 & -56/65\\ -56/65 & 33/65\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}-356/65\\ -92/65\end{bmatrix}$

6.11 Consider the lin transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector through an angle of $45°$ in the clockwise direction. $\varnothing = -\pi/4$

a) Find the matrix A of the transformation T. $\cos\varnothing = 1/\sqrt{2}$ $\sin\varnothing = -1/\sqrt{2}$

$\begin{bmatrix}\cos\varnothing & -\sin\varnothing\\ \sin\varnothing & \cos\varnothing\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

b) Find the vector produced by transforming $\begin{bmatrix}-2\\-5\end{bmatrix}$ under T.

$T\left(\begin{bmatrix}-2\\-5\end{bmatrix}\right) = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}-2\\-5\end{bmatrix} = \begin{bmatrix}-7/\sqrt{2}\\ -3/\sqrt{2}\end{bmatrix}$

6.10 To every linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ there is an associated 2×2 matrix.

a) Reflection about the line $y=x$ $\vec{a} = \begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$ $\text{Ref}_L = \begin{bmatrix}2a_1^2-1 & 2a_1a_2\\ 2a_1a_2 & 2a_2^2-1\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}$

b) CCW rotation by $\pi/2$ rad $\cos\varnothing = 0$ $\sin\varnothing = 1$ $\text{Rot}_{\pi/2} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$

c) The projection onto the x-axis given by $T(x,y) = (x,0)$ $\vec{a} = [1\ 0]$ $\text{proj}_L = \begin{bmatrix}1&0\\0&0\end{bmatrix}$

d) Reflection about the x-axis $\vec{a} = [1\ 0]$ $\text{Ref}_L = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

e) CW rotation by $\pi/2$ rad $\varnothing = -\pi/2$ $\cos\varnothing = 0, \sin\varnothing = -1$ $\text{Rot}_{\pi/2} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$

f) Reflection about the y-axis $\vec{a} = \begin{bmatrix}0\\1\end{bmatrix}$, $\text{Ref}_L = \begin{bmatrix}-1&0\\0&1\end{bmatrix}$

p.s.
6.12D

"reflect about the origin" $T(X_1, X_2) = (-X_1, -X_2)$

$\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

"reflect about $X=0$" $T(X_1,X_2) = (-X_1, X_2)$ $\begin{bmatrix}-1&0\\0&1\end{bmatrix}$

"reflect about $y=0$" $T(X_1,X_2) = (X_1,-X_2)$ $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

Compositions of Linear Transformations

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^n \to \mathbb{R}^k$

The composition $S \circ T$ is defined to be the transformation $\mathbb{R}^n \to \mathbb{R}^k$ by:

$$(S \circ T)(\vec{x}) = S(T(\vec{x}))$$

Net: $\mathbb{R}^n \to \mathbb{R}^k$

$\mathbb{R}^n \to \mathbb{R}^m$

"Apply T, then S" $\qquad \mathbb{R}^m \longrightarrow \mathbb{R}^k$

If T & S are both linear, $(S \circ T)$ is also linear

The transformations T & S can both be represented by matrices.

How are these matrices related to $(S \circ T)$?

Let $\vec{A} = T$ for $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then $(S \circ T)$ represented by

$\vec{B} = S$ for $S: \mathbb{R}^m \to \mathbb{R}^k$ $\qquad BA$

Composition of linear transformation $\Longleftrightarrow$ matrix mult of corresponding transf's.

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \qquad S(A\vec{x})$$
$$B(A\vec{x})$$
$$(BA)\vec{x}$$

$(BA)\vec{x} = ST\,\vec{x}$

e.g. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin trans. that first reflects a vector across the line L, followed by rotating vectors by $\pi/3$ clockwisely. Find the matrix rep of T.

$T = Rot_{-\pi/3} \circ Ref_L$

$$Rot(-\pi/3) = \begin{bmatrix} \cos(-\pi/3) & -\sin(\pi/3) \\ \sin(-\pi/3) & \cos(-\pi/3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \sqrt{3}/2 \\ -\sqrt{3}/2 & \frac{1}{2} \end{bmatrix} \qquad Ref_L = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & \frac{1}{2} \end{bmatrix}$$

$$T = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

WW 7.5 a) $S: \mathbb{R}^2 \to \mathbb{R}^2$, linear trans that first rotates points clockwise through 150° and then reflects points through the line $x_2 = -x_1$.

$150°\ CW = -150 \times \frac{\pi}{180°} = \frac{-5\pi}{6}$

$\sin(5\pi/6) = -1/2$

$\cos(5\pi/6) = -\sqrt{3}/2$

$$\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix} = T$$

unit vector on the line $[1/\sqrt{2}, -1/\sqrt{2}]$ $\quad Ref_L = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = S \qquad ST = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$

WW 7.5 b) $T: \mathbb{R}^2 \to \mathbb{R}^2$, linear trans that first reflects points through $x_2 = -x_1$ and then rotates points clockwise through 150°.

$$TS = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Why $(S \circ T)\vec{x}$ represented by ST?

Do T first

$T(\vec{x}) = y$ $\qquad$ $A\vec{x} = y$

Then, do S

$S(T(\vec{x}))$ $\qquad$ $By = z$

$B(A\vec{x}) = z$

$BA\vec{x} = z$

A corresponds to T

B corresponds to S

BA corresponds to $(S \circ T)$

ww7.6 Consider 3 linear transformations R, S, T so that

- R maps $\begin{bmatrix}1\\0\end{bmatrix}$ into $\begin{bmatrix}-1\\1\end{bmatrix}$, S maps $\begin{bmatrix}-1\\1\end{bmatrix}$ to $\begin{bmatrix}1\\2\end{bmatrix}$, T maps $\begin{bmatrix}1\\2\end{bmatrix}$ to $\begin{bmatrix}6\\2\end{bmatrix}$.
- R maps $\begin{bmatrix}0\\1\end{bmatrix}$ into $\begin{bmatrix}1\\-2\end{bmatrix}$, S maps $\begin{bmatrix}1\\-2\end{bmatrix}$ to $\begin{bmatrix}-3\\-3\end{bmatrix}$, T maps $\begin{bmatrix}-3\\-3\end{bmatrix}$ to $\begin{bmatrix}-12\\0\end{bmatrix}$

$\begin{bmatrix}-1 & 1\\1 & -2\end{bmatrix}$ is the matrix of the transformation R $\quad \begin{bmatrix}R(e_1) & R(e_2)\end{bmatrix}$

$\begin{bmatrix}1 & -3\\2 & -3\end{bmatrix}$ is the matrix of the transformation do R, then S

$\begin{bmatrix}S(R(e_1)) & S(R(e_2))\end{bmatrix}$ $\quad S \circ R$

$\begin{bmatrix}6 & -12\\2 & 0\end{bmatrix}$ is the matrix of the transformation do R, then S, then T

$\begin{bmatrix}T(S(R(e_1))) & T(S(R(e_2)))\end{bmatrix}$ $\quad T \circ S \circ R$

matrix representing R:

$$\begin{bmatrix}R(e_1) & R(e_2)\end{bmatrix} = \begin{bmatrix}-1 & 1\\1 & -2\end{bmatrix}$$

matrix representing S:

$$\begin{bmatrix}1\\0\end{bmatrix} = -2\begin{bmatrix}-1\\1\end{bmatrix} - \begin{bmatrix}1\\-2\end{bmatrix}$$

$$S\begin{bmatrix}1\\0\end{bmatrix} = -2S\begin{bmatrix}-1\\1\end{bmatrix} - S\begin{bmatrix}1\\-2\end{bmatrix} = -2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}-3\\-3\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = -\begin{bmatrix}-1\\1\end{bmatrix} - \begin{bmatrix}1\\-2\end{bmatrix}$$

$$S\begin{bmatrix}0\\1\end{bmatrix} = -S\begin{bmatrix}-1\\1\end{bmatrix} - S\begin{bmatrix}1\\-2\end{bmatrix} = -\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}-3\\-3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}S(e_1) & S(e_2)\end{bmatrix} = \begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}$$

matrix representing T:

$$\begin{bmatrix}1\\0\end{bmatrix} = -2/3\begin{bmatrix}-3\\-3\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}$$

$$T\begin{bmatrix}1\\0\end{bmatrix} = -2/3\,T\begin{bmatrix}-3\\-3\end{bmatrix} - T\begin{bmatrix}1\\2\end{bmatrix} = -2/3\begin{bmatrix}-12\\0\end{bmatrix} - \begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} + 1/3\begin{bmatrix}-3\\-3\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}6\\2\end{bmatrix} + \begin{bmatrix}-4\\0\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$$

$$\begin{bmatrix}T(e_1) & T(e_2)\end{bmatrix} = \begin{bmatrix}2 & 2\\-2 & 2\end{bmatrix}$$

$RS = [\,-2$

$RT = [\,2$

$ST = [\,1 \quad -3$

$SR = [$

$TS = [$

$TR = [$

Not necessary.

ww7.7 Let the linear transformations R, S, T be defined as follows:

$$\overset{3\times 2}{R}\left(\overset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}\right) = \overset{3\times 1}{\begin{bmatrix} -2x_1 + 2x_2 \\ 2x_1 - x_2 \\ 2x_1 - 2x_2 \end{bmatrix}} \quad \mathbb{R}^2 \to \mathbb{R}^3 \qquad R: \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix}$$

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 2x_2 \\ -2x_1 + x_2 \end{bmatrix} \quad \mathbb{R}^2 \to \mathbb{R}^2 \qquad S: \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 - 2x_4 \\ x_1 - 2x_2 + 2x_4 \end{bmatrix} \quad \mathbb{R}^4 \to \mathbb{R}^2 \qquad T: \begin{bmatrix} 1 & 0 & 1 & -2 \\ 1 & -2 & 0 & 2 \end{bmatrix}$$

Find the matrix ~~re~~representations of the following transformations

By MatLAB,

$R \circ S$
$$\begin{bmatrix} -8 & -6 \\ 6 & -5 \\ 8 & -6 \end{bmatrix}$$

$R \circ T$
$$\begin{bmatrix} 0 & -4 & -2 & 8 \\ 1 & 2 & 2 & -6 \\ 0 & 4 & 2 & -8 \end{bmatrix}$$

$S \circ S$
$$\begin{bmatrix} 8 & -6 \\ -6 & 5 \end{bmatrix}$$

$T \circ T$

undefined

$T \circ R$

undefined

$$S \circ T: \begin{bmatrix} 0 & 4 & 2 & -8 \\ -1 & -2 & -2 & 6 \end{bmatrix}$$