Matrix Operations

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Week 6, 115

A : m x n matrix
m rows, n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & & a_{2n} \\ a_{i1} & & & & a_{ij} & & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & & a_{m4} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} is the entry on the i^{th} row & j^{th} col

- **Addition** $A + B = C$

$c_{ij} = a_{ij} + b_{ij}$ e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 9 & 3 \\ 2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 11 & 6 \\ 6 & 6 & 13 \end{bmatrix}$

- **Scalar Multiplication** $sA = C$

$c_{ij} = s a_{ij}$ e.g. $2\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}$ $0\begin{bmatrix} 2 & 4 \\ 9 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- **Matrix Multiplication**

$AB = C$
$(m \times n)\ (n \times p)\quad (m \times p)$

$BA = D$
$(n \times p)\ (m \times n)\quad (n \times n)$

D only defined if $p = m$

defined so long as (cols in first matrix) = (rows in second matrix)

$(m \times \boxed{n)(n} \times p)$

$m \times p$ = size of resulting matrix

For $AB = C$, entries $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

i^{th} row, j^{th} col of C i^{th} row j^{th} column

Basically, $c_{ij} = a_{i\cdot} \cdot b_{\cdot j}$

dot product the i^{th} row vec in A with j^{th} col vec in B

e.g.

$$\underset{2\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} [1\,2\,3]\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} & [1\,2\,3]\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix} \\ [4\,5\,6]\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} & [4\,5\,6]\cdot\begin{bmatrix}-1\\1\\2\end{bmatrix} \end{bmatrix}} = \begin{bmatrix} 1 & 7 \\ 4 & 13 \end{bmatrix}$$

$$\underset{3\times2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}} \underset{2\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} [1\ -1]\cdot\begin{bmatrix}1\\4\end{bmatrix} & [1\ -1]\begin{bmatrix}2\\5\end{bmatrix} & [1\ -1]\begin{bmatrix}3\\6\end{bmatrix} \\ [0\ 1]\cdot\begin{bmatrix}1\\4\end{bmatrix} & [0\ 1]\begin{bmatrix}2\\5\end{bmatrix} & [0\ 1]\begin{bmatrix}3\\6\end{bmatrix} \\ [0\ 2]\cdot\begin{bmatrix}1\\4\end{bmatrix} & [0\ 2]\begin{bmatrix}2\\5\end{bmatrix} & [0\ 2]\begin{bmatrix}3\\6\end{bmatrix} \end{bmatrix}} = \begin{bmatrix} -3 & -3 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

e.g.

$$\underset{4\times2}{\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 9 \\ 1 & 6 \end{bmatrix}} \underset{2\times3}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}} = \underset{4\times3}{\begin{bmatrix} 5 & 11 & 17 \\ 6 & 12 & 18 \\ 20 & 42 & 64 \\ 13 & 27 & 41 \end{bmatrix}}$$

$$\underset{2\times3}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}} \underset{4\times2}{\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 9 \\ 1 & 6 \end{bmatrix}}$$ is not defined

e.g. $\underset{2\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} x+2y+3z \\ 4x+5y+6z \end{bmatrix}$

Given the system
$x + 2y + 3z = 7$
$4x + 5y + 6z = 8$
$\Rightarrow$

Matrix form: $A\vec{v} = \vec{s}$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

Augmented matrix form:

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & 8 \end{array}\right]$

Systems & Matrix ↑

(6.1) $A = \begin{bmatrix} -4 & -2 & 4 \\ 4 & -1 & -4 \\ 2 & 4 & -4 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 3 \\ 3 & -1 & 3 \end{bmatrix}$

$4A = \begin{bmatrix} -16 & -8 & 16 \\ 16 & -4 & -16 \\ 8 & 16 & -16 \end{bmatrix}$ $2A - 4B = \begin{bmatrix} -4 & -12 & -4 \\ 8 & -6 & -20 \\ -8 & 12 & -20 \end{bmatrix}$

(6.2) Solve for $\vec{x}$ if $\begin{bmatrix} -6 & 9 & -2 \\ 1 & 3 & 4 \end{bmatrix} = -5\vec{x} - 5\begin{bmatrix} -4 & 7 & 1 \\ -5 & 4 & -6 \end{bmatrix}$

$$\begin{bmatrix} 6 & -9 & 2 \\ -1 & -3 & -4 \end{bmatrix} = 5\vec{x} + \begin{bmatrix} -20 & 35 & 5 \\ -25 & 20 & -30 \end{bmatrix}$$

$$\begin{bmatrix} 26 & -44 & -3 \\ 24 & -23 & 26 \end{bmatrix} = 5\vec{x}$$

$$\vec{x} = \begin{bmatrix} 26/5 & -44/5 & -3/5 \\ 24/5 & -23/5 & 26/5 \end{bmatrix}$$

(6.3) Write the system $\begin{cases} -4y + 10z = a \\ 9x + 3y = b \\ -5x - 3y + 4z = c \end{cases}$

in matrix form $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 0 & -4 & 10 \\ 9 & 3 & 0 \\ -5 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Find $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 & -4 & 10 \\ 9 & 3 & 0 \\ -5 & -3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ -18 \\ 18 \end{bmatrix}$

Can do matrix multiplication, or plug into the system!

$a = -4(-3) + 10(1) = 22$

$b = 9(-1) + 3(-3) = -18$

$c = -5(-1) - 3(-3) + 4(1) = 18$

6.4 $\begin{bmatrix} -7 & 0 & 4 \\ -3 & 9 & -3 \\ -8 & 1 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \\ -9 \end{bmatrix} = \begin{bmatrix} -85 \\ -21 \\ -140 \end{bmatrix}$ $\quad \begin{bmatrix} -7 & 0 & 4 \\ -3 & 9 & -3 \\ -8 & 1 & 9 \end{bmatrix} \begin{bmatrix} -7 \\ 6 \\ -7 \end{bmatrix} = \begin{bmatrix} 21 \\ 96 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} -7 & 0 & 4 \\ -3 & 9 & -3 \\ -8 & 1 & 9 \end{bmatrix} \begin{bmatrix} 7 & -7 \\ -3 & 6 \\ -9 & -7 \end{bmatrix} = \begin{bmatrix} -85 & 21 \\ -21 & 96 \\ -140 & -1 \end{bmatrix}$$

6.5 $A = \begin{bmatrix} 3 & 4 & -2 \\ 3 & 4 & 4 \\ -2 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & -4 & -3 \\ 0 & 0 & 4 \\ 3 & -1 & -2 \end{bmatrix}$

$$AB = \begin{bmatrix} -6-6=-12 & -12+2=-10 & -9+16+4=11 \\ -6+12=6 & -12-4=-16 & -9+16-8=-1 \\ 4+6=10 & 8-2=6 & 6+12-4=14 \end{bmatrix} = \begin{bmatrix} -12 & -10 & 11 \\ 6 & -16 & -1 \\ 10 & 6 & 14 \end{bmatrix}$$

$$BA = \begin{bmatrix} -6-12+6=-12 & -8-16-9=-33 & 4-16-6=-18 \\ -8 & 12 & 8 \\ 9-3+4=10 & 12-4-6=2 & -6-4-4=-14 \end{bmatrix} \begin{bmatrix} -12 & -33 & -18 \\ -8 & 12 & 8 \\ 10 & 2 & -14 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 3 & 4 & -2 \\ 3 & 4 & 4 \\ -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 & -2 \\ 3 & 4 & 4 \\ -2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 9+12+4=25 & 22 & 6 \\ 9+12-8=13 & 40 & 18 \\ -6+9-4=-1 & 10 & 20 \end{bmatrix}$$

6.6 $A = \begin{bmatrix} 1 & -2 & -3 \\ -1 & -3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 1 \\ 3 & -3 \\ -2 & -1 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 2$

2×2

$$AB = \begin{bmatrix} -3-6+6=-3 & 1+6+3=10 \\ 3-9-6=-12 & -1+9-3=5 \end{bmatrix}$$

3×3

$$BA = \begin{bmatrix} -3-1=-4 & 6-3=3 & 9+3=12 \\ 3+3=6 & -6+9=3 & -9-9=-18 \\ -2+1=-1 & 4+3=7 & 6-3=3 \end{bmatrix}$$

(6.7) Compute $\begin{bmatrix} -4 & -4 \\ -1 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & -4 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -12-4+2=-14 & -4-4+4=-4 \\ -3-4+1=-6 & -1-4-4=-9 \\ -9+1-4=-12 & -3+1+1=-1 \end{bmatrix}$

(6.8) Let $A = \begin{bmatrix} x & -7 \\ y & -5 \end{bmatrix}$ Determine the values for x and y for which $A^2 = A$

$$A^2 = \begin{bmatrix} x & -7 \\ y & -5 \end{bmatrix}\begin{bmatrix} x & -7 \\ y & -5 \end{bmatrix} = \begin{bmatrix} x^2-7y & -7x+35 \\ xy-5y & -7y+25 \end{bmatrix}$$

$x = x^2 - 7y$ $\quad -7 = -7x + 35 \quad x = 6$

$y = xy - 5y$ $\quad -5 = -7y + 25 \quad y = 30/7$

(6.9) If A, B, C are 4x4, 4x7, 7x9 respectively, determine which are defined. If defined, what is the size of the resulting matrix?

3C	$3[7\times9]$	7×9
BC	$[4\times\boxed{7}][\boxed{7}\times9]$	4×9
CB	$[7\times\boxed{9}][\boxed{4}\times7]$	undefined
B^2	$[4\times\boxed{7}][\boxed{4}\times7]$	undefined

Cool Matrices

- Zeros Matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- Identity Matrix I_n $(n \times n)$ $a_{ij} = 1, \; i = j$; $a_{ij} = 0 \; i \neq j$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$
- Diagonal: non-diagonal entries $= 0$

  diagonal $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ non-diagonal $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$
- Upper, Lower Triangular

  square matrices with entries below, above (respectively) diagonal $= 0$.

  $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ upper Δr $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 9 & 0 \\ 4 & 8 & 6 \end{bmatrix}$ lower Δr $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is both upper & lower triangular.

Matrix Operations Properties

① $A + B = B + A$

② $A + B + C = A + (B + C) = (A + B) + C$

③ $s(A+B) = sA + sB$

$(s+t)A = sA + tA$

④ $(st)A = (tA)s$

⑤ $1A = A$

$0A = 0_{matrix}$

Rmk $AB \neq BA$

$(A+B)^2$, A, B are $n \times n$

$= (A+B)(A+B)$

$= A(A+B) + B(A+B)$

$= A^2 + AB + BA + B^2$

⑥ $A(B+C) = AB + AC$

* ⑦ $A(BC) = (AB)C$

⑧ $s(AB) = (sA)B = A(sB)$

⑨ If A is an $m \times n$ matrix, whenever it is defined,

$A = I_m A$ $(m \times m)(m \times n) \to m \times n$

$A = A I_n$ $(m \times n)(n \times n) \to m \times n$

e.g. $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2 \times 2} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2 \times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Transpose

The *transpose* of an $m \times n$ matrix A is an $n \times m$ matrix B with $b_{ij} = a_{ji}$

$$A^T = B$$

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ e.g. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

Property: Let $C: m \times n$, $D: n \times k$ matrices

$$\underbrace{[\underbrace{CD}_{m \times k}]^T}_{k \times m} = \underbrace{\underset{k \times n}{D^T} \; \underset{n \times m}{C^T}}_{k \times m}$$

$\boxed{\text{e.g.}}$

$$\underset{2 \times 3}{\overset{C}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}} \underset{3 \times 1}{\overset{D}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}} = \underset{2 \times 1}{\overset{CD}{\begin{bmatrix} 6 \\ 15 \end{bmatrix}}}$$

$$\underset{\substack{1 \times 3 \\ D^T}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}} \cdot \underset{\substack{3 \times 2 \\ C^T}}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} = \underset{\substack{1 \times 2 \\ [CD]^T}}{\begin{bmatrix} 6 & 15 \end{bmatrix}}$$

$\boxed{\text{e.g.}}$ Let $\vec{x}, \vec{y}$ be column vectors in $\mathbb{R}^n$. Take dot/inner product.

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \dots x_n y_n = x^T y$$

$$\underset{n \times 1}{[\;]} \underset{n \times 1}{[\;]} \qquad \underset{1 \times n}{[\quad]} \underset{n \times 1}{[\;]} = \underset{1 \times 1}{[\;]}$$

$$= y^T x$$

$$\underset{1 \times n}{[\quad]} \underset{n \times 1}{[\;]} = \underset{1 \times 1}{[\;]}$$

$\boxed{7.14}$ $(A+B)^T = A^T + B^T$.

The transpose of a sum of matrices equals the sum of their transposes

$\boxed{7.14}$ If A is $n \times n$, $(A^2)^T = (A^T)^2$ $\qquad (AA)^T = A^T A^T = (A^T)^2$

$\boxed{7.15}$ $A: 3 \times 3$
$B: 3 \times 8$
$C: 8 \times 6$

$5C^T: 6 \times 8$ $\qquad CC^T: 8 \times 8$
$A + B^T$: undefined
$A^T B^T$: undefined

$\boxed{7.16}$ $\vec{u} = \begin{bmatrix} -8 \\ -4 \\ -4 \end{bmatrix}$

$$uu^T = \begin{bmatrix} -8 \\ -4 \\ -4 \end{bmatrix} \begin{bmatrix} -8 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 64 & 32 & 32 \\ 32 & 16 & 16 \\ 32 & 16 & 16 \end{bmatrix}$$

$$u^T u = \begin{bmatrix} -8 & -4 & -4 \end{bmatrix} \begin{bmatrix} -8 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 96 \end{bmatrix}$$