Homogeneous Systems

- A linear system is homogenous if all constant terms are zero.

e.g. $x + 2y + z = 0$, $x + 3y + 2z = 0$ non-e.g. $x + 2y + z = 1$, $x + 3y + 2z = 2$

Properties

① $(0, 0, 0, \ldots, 0)$ is a solution

② If $\vec{x}, \vec{y}$ are solutions, then $\vec{x} + \vec{y}$ is another solution.
$k\vec{x}$ is also a solution (k is a scalar)

e.g. $[1, -1, 1], [2, -2, 2]$ are solutions to $x + 2y + z = 0$

Then, $[1, -1, 1] + [2, -2, 2] = [3, -3, 3]$ is another solution
$(-5)[1, -1, 1]$ is also a solution

linear system $\xrightarrow[\text{to zero}]{\text{change all constant terms}}$ associated homogenous system

linear system

(*)
$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{array}\right] \end{array}$$

G.E. ↓

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$x + y + s = 2$

$y = 1$

$x + 1 + s = 2$

$x = 1 - s$

$s = s$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-s \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

a particular solution of (*)   solution of (**)

associated homogenous system

(**)
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right]$$

G.E. ↓

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$x + y + s = 0$

$y = 0$

$x = -s$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

solution of (**)

Solution of a linear system
= a particular solution (*)
+
solution of the associated homogenous system (**)

E.g.

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ -1 & -1 & 1 & -2 & -1 \\ 1 & 1 & 3 & -2 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$y = s \qquad w = t$

$z = 1 + t$

$x = 3 - s - (1+t)$

$= 2 - s - t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

B

ass't homogenous system

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & -2 & 0 \\ 1 & 1 & 3 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[\text{operations}]{\text{same}} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{2} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

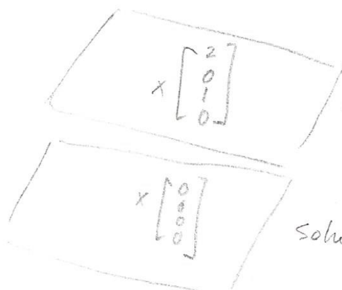
$y = s \qquad w = t$

$z = t$

$x = -s - (t)$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

A



solution of B

solution of A

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & -2 & 1 \\ 1 & 1 & 3 & -2 & 3 \end{array}\right]$$

same

$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is a particular solution

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

ww5.3 Find the general sol'n to the homogeneous system

$$\begin{aligned} x - y - z + 4w &= 0 \\ 3x + 2y - z + 3w &= 0 \end{aligned}$$

$$[x, y, z, w] = s\left[\tfrac{3}{5}, \tfrac{-2}{5}, 1, 0\right] + t\left[\tfrac{-11}{5}, \tfrac{9}{5}, 0, 1\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 4 & 0 \\ 3 & 2 & -1 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 4 & 0 \\ 0 & 5 & 2 & -9 & 0 \end{array}\right]$$

$$5y = 9t - 2s \qquad x = \left(\tfrac{9}{5}t - \tfrac{2}{5}s\right) + s - 4t$$

$$y = \tfrac{9}{5}t - \tfrac{2}{5}s \qquad = \tfrac{-11}{5}t + \tfrac{3}{5}s$$

ww. 5.4 Find the general sol'n of the homogeneous system

$$A = \begin{array}{c} \left[\begin{array}{cccccc|c} \boxed{1} & 1 & 3 & 1 & 4 & 5 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} a & s_1 & b & s_2 & c & s_3 \end{array} \end{array}$$

$$\begin{aligned} \vec{x} = & \; s_1 [-1, 1, 0, 0, 0, 0] \\ & + s_2 [8, 0, -3, 1, 0, 0] \\ & + s_3 [14, 0, -1, 0, -4, 1] \end{aligned}$$

where s_1, s_2, s_3 are free.

$$\begin{aligned} c &= -4s_3 \\ b &= -3s^2 - s_3 \\ a &= -s_1 - 3b - s_2 - 4c - 5s_3 \\ &= -s_1 - 3(-3s^2 - s_3) - s_2 - 4(-4s_3) - 5s_3 \\ &= -s_1 + 8s_2 + 14s_3 \end{aligned}$$

$$\begin{bmatrix} a \\ s_1 \\ b \\ s_2 \\ c \\ s_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} s_1 + \begin{bmatrix} 8 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s_2 + \begin{bmatrix} 14 \\ 0 \\ -1 \\ 0 \\ -4 \\ 1 \end{bmatrix} s_3$$

WW 5.5 Consider the chemical reaction

$$\overset{6}{a}\ C_2H_6 + \overset{2}{b}\ CO_2 + \overset{3}{c}\ H_2O \rightarrow \overset{7}{d}\ C_2H_5OH$$

a, b, c, d are unknown integers > 0.
The reaction must be balanced.
# of atoms of each element must be the same before & after the rxn.
Eg. B/c # O atoms remains:

$$2b + c = d$$

It is customary to use smallest integers > 0.
Balance.

Note: Such rxn balance problems are always equivalent to a homogeneous system w/ a single free parameter.

balance C's : $2a + b = 2d$
H's : $6a + 2c = 6d$
O's : $2b + c = d$

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 3 & 0 & 1 & -3 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 0 & -3 & 2 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 7 & -3 & 0 \end{array}\right]$$

s

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s \begin{bmatrix} 6 \\ 2 \\ 3 \\ 7 \end{bmatrix}$$

$7c = 3s$ $\quad c = 3/7\, s$

$3b = 2(3/7\, s)$ $\quad b = 2/7\, s$

$2a = 2s - (7/7\, s)$ $\quad a = 6/7\, s$

$d = 1$

$\times 7$

Relationship between Linear Systems, Linear Combinations, Linear Independence

e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

① Is $\vec{w}$ a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + u\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s_1 - s_2 + 3s_3 \\ -s_1 + 2s_2 \\ s_1 + s_3 \\ s_1 - s_2 \end{bmatrix}$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \\ \begin{array}{cccc} v_1 & v_2 & v_3 & w \end{array} \end{array} \xrightarrow[\text{row ops}]{\text{lots of}} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \leftarrow \text{impossible} \\ \text{ref} \end{array}$$

$\Rightarrow$ No solution to the linear system

$\Rightarrow$ $\vec{w}$ is *not* a linear combination of v_1, v_2, v_3

② Is $\{v_1, v_2, v_3\}$ linearly independent?

Suppose $s_1\vec{v}_1 + s_2\vec{v}_2 + s_3\vec{v}_3 = \vec{0}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = s_1\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + s_2\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s_3\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} v_1 & v_2 & v_3 & \vec{0} \end{array} \end{array} \xrightarrow[\text{see ①}]{} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

No free variables $\Rightarrow$ unique sol'n

$s_1 = s_2 = s_3$

$\Rightarrow \{v_1, v_2, v_3\}$ is linearly independent

Lessons

① If there is a solution to the linear system, then it is a linear combination

② If the only solution is $(0, 0, \ldots 0)$, it is linearly independent.

Linear Combinations, Linear Independence via Linear Systems

Rmk: In both (*) and (**), vectors should be written vertically

Given vectors $\vec{v_1}, \vec{v_2}, \dots \vec{v_k}, \vec{w}$ in $\mathbb{R}^n$

- $\{\vec{v_1}, \vec{v_2}, \dots \vec{v_k}\}$ is lin independent

$\iff s_1\vec{v_1} + s_2\vec{v_2} + \dots + s_k\vec{v_k} = \vec{0}$ and all $s_i = 0$ is the only sol'n

$$\iff \begin{array}{cccc} v_1 & v_2 & \dots & v_k \end{array} \quad \left[\begin{array}{cccc|c} | & | & & | & 0 \\ | & | & \dots & | & 0 \\ | & | & & | & \vdots \\ | & | & & | & 0 \end{array}\right] \quad (*)$$

has the unique sol'n $s_1 = s_2 = s_i = s_k = 0$

$\iff$ ref of (*) has no free vars

Note: the sys always has at least one sol'n $\vec{s} = \vec{0} = (0, 0, \dots 0)$

- $\vec{w}$ is a linear combination of $\vec{v_1}, \vec{v_2} \dots \vec{v_k}$

$\iff s_1\vec{v_1} + s_2\vec{v_2} + \dots + s_k\vec{v_k} = \vec{w}$ for some $s_1, s_2, \dots s_k$

$$\iff \begin{array}{cccc} v_1 & v_2 & \dots & v_k \end{array} \quad \left[\begin{array}{cccc|c} | & | & & | & | \\ | & | & \dots & | & w \\ | & | & & | & | \end{array}\right] \quad (**)$$

has at least one solution

$\iff$ ref of (**) has no row with

$[\,0 \;\; 0 \;\; 0 \;\; \dots \;\; 0 \mid * \,]$

$\uparrow$ non-zero

Eg. $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ $\quad \vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ $\quad \vec{v_3} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -5 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}$

Show that $\vec{w}$ is a linear combination of v_1, v_2, v_3 and give one such expression

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} = s_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + s_3 \begin{bmatrix} 3 \\ -2 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} s_1 - s_2 + 3s_3 \\ -s_1 + 2s_2 - 2s_3 \\ s_1 \quad + 4s_3 \\ -s_1 - s_2 - 5s_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ -1 & 2 & -2 & 1 \\ 1 & 0 & 4 & 1 \\ -1 & -1 & -5 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (*)$$

ref

No contradiction

$\Rightarrow$ at least one sol'n

$\therefore \vec{w}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$

To express $\vec{w}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$, Find sol'n to *

Let $s_3 = t$

$s_2 + t = 1$

$s_2 = 1 - t$

$s_1 - s_2 + 3t = 0$

$s_1 = s_2 - 3t$

$s_1 = 1 - 4t$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1-4t \\ 1-t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$$

$t = 0$ (or any #) $\Rightarrow$ a particular sol'n $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{w} = (1)\vec{v_1} + (1)\vec{v_2} + (0)\vec{v_3}$$

E.g. Show that $\{v_1, v_2, v_3\}$ is linearly dependent and give a non trivial relation b/w them

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 5 \end{bmatrix}$$

Suppose that $s_1\vec{v}_1 + s_2\vec{v}_2 + s_3\vec{v}_3 = \vec{0}$

Then,

$$(**) \quad \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ -1 & 2 & -2 & 0 \\ 1 & 6 & 4 & 0 \\ -1 & -1 & -5 & 0 \end{array}\right] \xrightarrow[\text{back of this pg}]{\text{from sol'n of}} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 3 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Finding a non-trivial linear relation is finding a non-zero sol'n to $(**)$

s_3 is a free variable

$\Rightarrow$ there is a non-zero sol'n

$\Rightarrow$ linearly dependent

From sol'n of back page or from $(**)$: $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = t\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$

Put $t = 1$, a non-zero sol'n $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$

$$(-4)\vec{v}_1 + (-1)\vec{v}_2 + (1)\vec{v}_3 = \vec{0}$$