Linear Systems

① Solve $\begin{cases} x+y = 4 & (A) \\ 2x-y = -1 & (B) \end{cases}$

Express y or x in terms of one another

(A) => $y = 4-x$ (C)

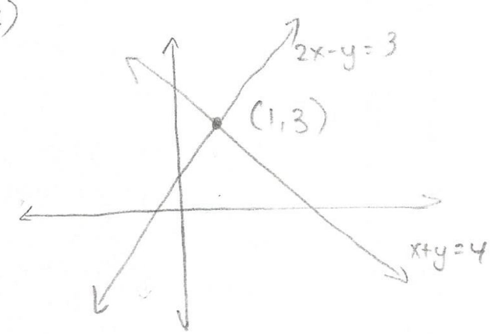
(C) => (B) $2x-(4-x) = -1$

$3x = 3$

$x = 1$

=> (C) $y = 3$

$(x,y) = (1,3)$



② $\begin{cases} x+y = 4 & (A) \\ 2x+2y = 7 & (B) \end{cases}$

(A) => $y = 4-x$ (C)

(C) => (B) $2x+2(4-x) = 7$

$2+8-2x = 7$

$8 = 7$

impossible

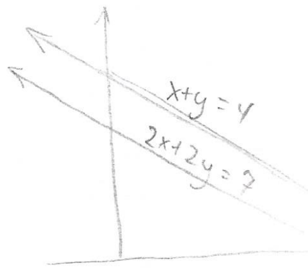
(A) => $x = 4-y$ (C)

(C) => (B) $2(4-y)+2y = 7$

$8-2y+2y = 7$

$8 = 7$

Impossible



No solution

Case 1: a solution : the 2 lines are not parallel

Case 2: no solutions : they are parallel

Case 3: many/all sol'ns : they are the same line.

If they are not parallel, there is at least one solution ($\mathbb{R}^2$)

WW3 (6) Solve the following system

$$\begin{cases} 6x + 9y = -30 & (A) \\ -5x + 7y = -120 & (B) \end{cases}$$

(A) $6x = -30 - 9y$

$x = -5 - \frac{3}{2}y$

$\Rightarrow$ (B) $-5(-5 - \frac{3}{2}y) + 7y = -120$

$25 + 7.5y + 7y = -120$

$14.5y = -145$

$y = -10$

$x = -5 - \frac{3}{2}(-10)$

$= 10$

$(x, y) = (10, -10)$

WW3 (7) Solve the system.

$$4x + 5y = a \quad (A)$$
$$-5x - 6y = b \quad (B)$$

(A) $4x = a - 5y$

$x = \frac{1}{4}a - \frac{5}{4}y$

$\Rightarrow$ (B) $-5(\frac{1}{4}a - \frac{5}{4}y) - 6y = b$

$-\frac{5}{4}a + \frac{25}{4}y - 6y = b$

$\frac{1}{4}y = b + \frac{5}{4}a$

$y = 4b + 5a$

$x = \frac{1}{4}a - \frac{5}{4}(4b + 5a)$

$= \frac{1}{4}a - 5b - \frac{25}{4}a$

$= -5b - 6a$

$(x, y) = (-5b - 6a, 4b + 5a)$

WW3 8) Find the quadratic polynomial whose graph goes through the points $(-1,5)$, $(0,4)$, and $(2,20)$.

$f(x) = \_x^2 + \_x + \_$

$a(-1)^2 + b(-1) + c = 5$

$c = 4$

$a(2)^2 + b(2) + c = 20$

$a - b = 1$

$4a + 2b = 16$

$a = b + 1$

$4(b+1) + 2b = 16$

$4b + 4 + 2b = 16$

$6b = 12$

$b = 2$

$a = (2) + 1$

$= 3$

$f(x) = 3x^2 + 2x + 4$

Geometry of Solutions of Linear System in $\mathbb{R}^2, \mathbb{R}^3$

① 2 variables x, y

Each equation $b_1x + b_2y = c$ defines a line in $\mathbb{R}^2$ (if $\vec{b} = [b_1, b_2] \neq \vec{0}$)

Solution of a linear system $\longleftrightarrow$ Common intersection of the corresponding lines

Special case: ⊛ $\begin{cases} b_{11}x + b_{12}y = c_1 & \longleftrightarrow L_1 \\ b_{21}x + b_{22}y = c_2 & \longleftrightarrow L_2 \end{cases}$

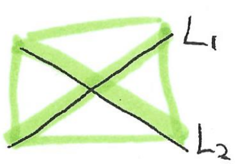
The system ⊛ has unique solution $\iff$ L_1, L_2 have unique intersection

$\iff$ L_1, L_2 are not parallel

$\iff$ Their normal vectors $\vec{b}_1 = [b_{11}, b_{12}]$, $\vec{b}_2 = [b_{21}, b_{22}]$ are not collinear

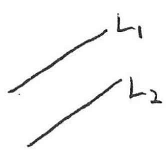
$\iff$ $\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \neq 0$

det $\neq 0$



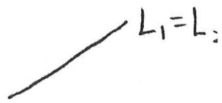
unique solution

det $= 0$



no solution

or



infinitely many solutions

e.g.

Does $\begin{cases} 2x - y = 1 \\ x - 3y = 2 \end{cases}$ have a unique sol'n?

$\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -5$ $\quad -5 \neq 0$; unique sol'n

② 3 variables x, y, z (if $\vec{b} = [b_1, b_2, b_3] \neq \vec{0}$)

Each equation $b_1x + b_2y + b_3z = c$ defines a plane in $\mathbb{R}^3$

Solution of a linear system $\longleftrightarrow$ Common intersection of the corresponding planes.

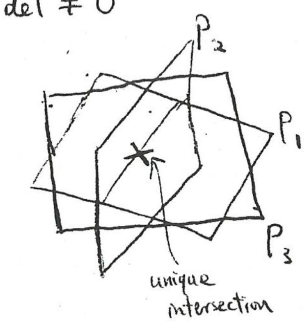
Special Case: ⊛⊛ $\begin{cases} b_{11}x + b_{12}y + b_{13}z = c_1 & \longleftrightarrow P_1 \\ b_{21}x + b_{22}y + b_{23}z = c_2 & \longleftrightarrow P_2 \\ b_{31}x + b_{32}y + b_{33}z = c_3 & \longleftrightarrow P_3 \end{cases}$

The system ⊛⊛ has unique solution $\iff$ P_1, P_2, P_3 have unique common intersection

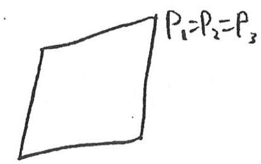
$\iff$ Their normal vectors $\vec{b}_1 = [b_{11}, b_{12}, b_{13}]$, $\vec{b}_2 = [b_{21}, b_{22}, b_{23}]$, $\vec{b}_3 = [b_{31}, b_{32}, b_{33}]$ are not coplanar i.e. (do not lie on the same plane)

$\iff$ $\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} \neq 0$

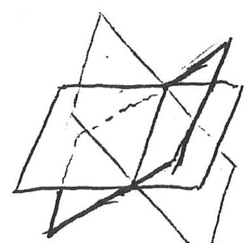
det $\neq 0$



det $= 0$



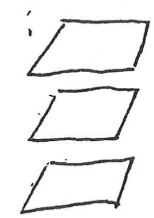
Intersection is a plane

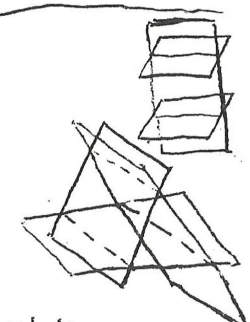


Intersection is a line

more than one sol'n

solution is a line.

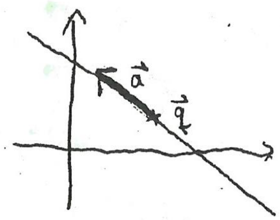




no solution

Summary: Parametric and Equation of Lines in $\mathbb{R}^2$

Given a point $\vec{q}$ on a straight line L and a direction $\vec{a}$ of L

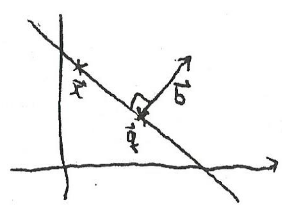


Parametric form of the line is

$$\vec{x} = \vec{q} + s\vec{a}$$

i.e. $[x_1, x_2] = [q_1 + sa_1, q_2 + sa_2]$

Given a point $\vec{q}$ on a straight line L and a normal direction $\vec{b}$ of L



For any point $\vec{x} = [x_1, x_2]$ on L

$$(\vec{x} - \vec{q}) \cdot \vec{b} = 0$$

$$[x_1 - q_1, x_2 - q_2] \cdot [b_1, b_2] = 0$$

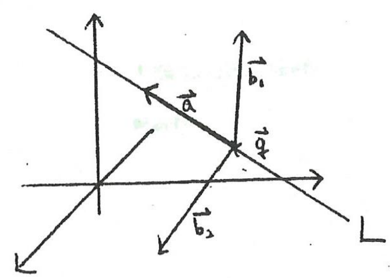
$$b_1(x_1 - q_1) + b_2(x_2 - q_2) = 0$$

∴ Equation form of L is

$$b_1x_1 + b_2x_2 = c, \text{ where } c = \vec{b} \cdot \vec{q} = b_1q_1 + b_2q_2$$

Summary: Parametric and Equation form of Lines and Planes in $\mathbb{R}^3$

Line L



- $\vec{q} = [q_1, q_2, q_3]$ is a point on L
- $\vec{a} = [a_1, a_2, a_3]$ is a direction of L
- $\vec{b_1} = [b_{11}, b_{12}, b_{13}]$, $\vec{b_2} = [b_{21}, b_{22}, b_{23}]$ are two (non-zero) non-collinear vectors orthogonal (perpendicular) to L

To describe points $\vec{x} = [x_1, x_2, x_3]$ on L,

Parametric form

$$\vec{x} = \vec{q} + s\vec{a}$$

i.e. $[x_1, x_2, x_3] = [q_1 + sa_1, q_2 + sa_2, q_3 + sa_3]$

Equation form

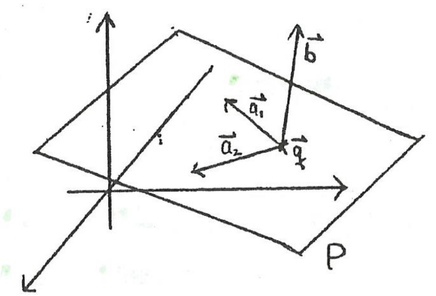
$$\begin{cases} (\vec{x} - \vec{q}) \cdot \vec{b_1} = 0 \\ (\vec{x} - \vec{q}) \cdot \vec{b_2} = 0 \end{cases}$$

i.e.

$$\begin{cases} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 = c_1 \\ b_{21}x_1 + b_{22}x_2 + b_{23}x_3 = c_2 \end{cases}$$

where $c_1 = \vec{b_1} \cdot \vec{q}$, $c_2 = \vec{b_2} \cdot \vec{q}$

Plane P



- $\vec{q} = [q_1, q_2, q_3]$ is a point on P
- $\vec{a_1} = [a_{11}, a_{12}, a_{13}]$, $\vec{a_2} = [a_{21}, a_{22}, a_{23}]$ are two (non-zero) non-collinear vectors in the direction of P.
- $\vec{b} = [b_1, b_2, b_3]$ is orthogonal to L

To describe points $\vec{x} = [x_1, x_2, x_3]$ on P,

Parametric form

$$\vec{x} = \vec{q} + s\vec{a_1} + t\vec{a_2}$$

i.e. $[x_1, x_2, x_3] = [q_1 + sa_{11} + ta_{21}, q_2 + sa_{12} + ta_{22}, q_3 + sa_{13} + ta_{23}]$

Equation form

$$(\vec{x} - \vec{q}) \cdot \vec{b} = 0$$

i.e. $b_1x_1 + b_2x_2 + b_3x_3 = c$

where $c = \vec{b} \cdot \vec{q}$

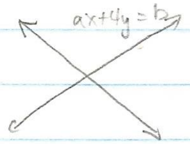
e.g. Does $\begin{cases} 2x - y = 1 \\ x - 3y = 2 \end{cases}$ have a unique sol'n? $\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} \neq 0$; Yes

e.g. Does $\begin{cases} x + 2y + 3z = 1 \\ 4x + 5y + 6z = 2 \\ 7x + 8y + 9z = 3 \end{cases}$ have a unique sol'n? $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (45-48) - 2(36-42) + 3(32-35) = 0$. No.

e.g. For what values of a & b such that the system given by 2 eq'ns and 2 unknowns: $\begin{matrix} x - 2y = 1 \\ ax + 4y = b \end{matrix}$ have a unique sol'n.

$\begin{vmatrix} 1 & -2 \\ a & 4 \end{vmatrix} = 4 + 2a$ Unique sol'n when $4 + 2a \neq 0$

$a \neq -2$



b moves the line up and down (like intercept)

If $a = -2$, the system has
- no solution (if intercept different)
- infinite solutions (if intercept same), depending on b.

ww.4 (2) Let r and s be unknown constants. Consider the linear system

$$-6x + 7y = r$$
$$3x + sy = -8$$

$\begin{vmatrix} -6 & 7 \\ 3 & s \end{vmatrix} = -6s - 21$ Unique when $-6s - 21 \neq 0$, $6s \neq -21$, $s \neq -21/6$

This system has a unique sol'n when s is not equal to $-21/6$.

If the above condition does not hold, ... $s = {}^{21}/6$, the 2 lines have the same slope. b determines the position of one line.

then the system will have no sol'ns when $r \neq 16$

and infinitely many sol'ns when $r = 16$

$7y = r + 6x$

$y = \frac{1}{7}r + \frac{6}{7}x$

$3x + \left(\frac{-21}{6}\right)y = -8$

$\frac{21}{6}y = 3x + 8$

$y = \frac{6}{7}x + \frac{48}{21}$

Same intercept when $\left(\frac{1}{7}\right)r = \frac{48}{21}$, $r = 16$

$\therefore$ Infinite sol'ns when $r = 16$. Same line

No solution when $r \neq 16$: Parallel & same slope.

Rank of an (~~Augmented~~) Matrix

Given a matrix A, rref of A is unique, but ref A is not.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

both ref

rref

Any two refs of A have same number of non-zero rows.

(ref has same # of non-zero rows as rref)

Rank(A): number of NONZERO rows in ref of A

= number of pivots in ref(A).

Note: For an $n \times m$ matrix, $\text{Rank } A \leq n, m$

The # of non-zero rows in ref(A) $\leq$ # rows of $A = n$

Each column has at most one pivot $\Rightarrow \text{rank } A \leq m$

e.g. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ Rank = 3

ref

e.g. $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Rank = 2

not ref

ref

Lecture 12

- Gaussian Elimination

Last time: (Reduced) Row echelon form.

> In ref (or rref)
>
> The leading non-zero entry on each row is called a pivot
>
> The corresponding variables are called basic variables
>
> Other variables are called free variables

eg. $\begin{cases} x_1+2x_2+3x_3+4x_4+5x_5=-1 \\ 6x_3+7x_4+8x_5=-2 \\ 9x_4+10x_5=-3 \end{cases} \leftrightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \underbrace{\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 4 & 5 & -1 \\ 0 & 0 & \boxed{6} & 7 & 8 & -2 \\ 0 & 0 & 0 & \boxed{9} & 10 & -3 \end{array}\right]}_{\text{already in ref}} \end{array}$

$\therefore$ x_1, x_3, x_4 are basic variables $\square$ 1, 6, 9 are pivots

x_2, x_5 are free variables

Solving Linear system by Gaussian Elimination

① Write down augmented matrix

② Put it into ref (or rref) by elementary row operations.

③ Any contradiction? (like $[0\ 0\ 0\ 0 \mid 1]$)

a. Yes $\Rightarrow$ No solution

b. No $\Rightarrow$ Unique solution if no free variables

Infinitely many solution if there is a free variables

Introduce a parameter to each free variable and write down solution (in terms of parameters) by backward substitution e.g. $\begin{cases} x+y=3 \\ -x+2y=1 \\ 3x \quad =7 \end{cases}$ Augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 0 & 7 \end{array}\right]$

Elementary row operations to (augmented) matrices

① Multiply a row by a <u>non-zero</u> constant

② Adding a multiple of a row to another row

③ Interchanging two rows

e.g. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \end{array}\right]$

$R_1 \times 3$: $\left[\begin{array}{ccc|c} 3 & 6 & 9 & 3 \\ 4 & 5 & 6 & 2 \end{array}\right]$

$R_2 \to R_2 + 2R_1$: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 6 & 9 & 12 & 4 \end{array}\right]$

$R_1 \leftrightarrow R_2$: $\left[\begin{array}{ccc|c} 4 & 5 & 6 & 2 \\ 1 & 2 & 3 & 1 \end{array}\right]$

<u>Defn</u> An augmented matrix is in **<u>row echelon form</u>** (ref) if

① All the **zero rows** (if exists) are at the **bottom**

② The **first non-zero entry** in each row is on the **right** of the ones in the rows above

$\begin{matrix} 1 & & \\ 0 & 2 & \\ 0 & 0 & 3 \end{matrix}$

It is called **<u>reduced row echelon form</u>** (rref) if furthermore

③ The **first non-zero entry** in each row is **1**, and is the **<u>only non-zero entry in that column</u>**

e.g.

Not ref $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 1 & 1 & 0 & -2 \end{array}\right]$

× ① non-zero × ② not zero

ref but not rref $\left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{2} & 4 \end{array}\right]$ not 1 $\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 7 \\ 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{cccc|c} \boxed{1} & 1 & 2 & 4 & -2 \\ 0 & \boxed{1} & 0 & 1 & 2 \\ 0 & 0 & \boxed{-1} & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ not 1

rref $\left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 2 \end{array}\right]$ $\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Consider solving

$$\begin{cases} 2x + y = 1 & (1) \\ 2x + y + z = 2 & (2) \\ y + 2z = 3 & (3) \\ x + 2y + 2z = 4 & (4) \end{cases}$$

(2)-(1) $z = 2 - 1 = 1$

(4)-(3) $x = 4 - 3 = 1$

$2(1) + y = 1$
$y = -1$

$(-1) + 2(1) \neq 3$!

System has no solution. We need a systematic way to analyze & solve: Gaussian Elimination

Augmented Matrices

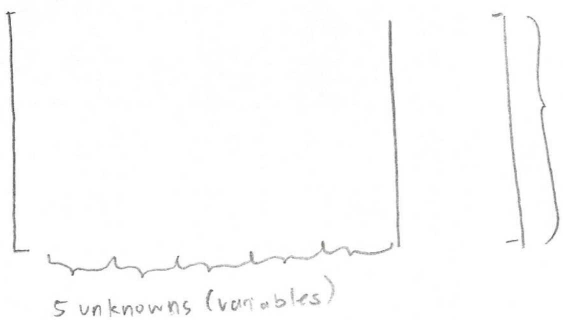
$$\begin{aligned} 2x + y + z &= 1 \\ -x + 3y - 2z &= 2 \end{aligned}$$

augmented matrix of the system

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ -1 & 3 & -2 & 2 \end{array}\right]$$

e.g. Given a linear system of 6 equations and 5 unknowns,

see Rank →



Find the # of solutions if...

	Rank of Unaugmented Matrix	Rank of Augmented Matrix	# of solutions
1)	4	5	0, one entry is non-zero [0 0 0 0 0 0 | x]
2)	5	5	1, No free, No contradiction
3)	2	2	infinite, 3 free variables

Gaussian Elimination Examples

(1) Unique Solution (2) No solution (3) Infinitely Many

(1) Unique

$$\begin{cases} x+y+z=3 \\ 3x+y-z=3 \\ -x-3y+z=-3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 1 & -1 & 3 \\ -1 & -3 & 1 & -3 \end{array}\right] \quad R_2 \to R_2 - 3R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -4 & -6 \\ -1 & -3 & 1 & -3 \end{array}\right]$$

$$R_3 \to R_1 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -4 & -6 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & 6 & 6 \end{array}\right] \quad \text{ref.}$$

Last Row: $0x + 0y + 6z = 6$

$\boxed{z=1}$

Second Row: $-2y - 4z = -6$

$-2y - 4 = -6$

$\boxed{y=1}$

First Row: $x+y+z=3$

$\Rightarrow x+1+1=3 \quad \boxed{x=1}$

Unique sol'n: $(x,y,z) = (1,1,1)$

(2) No

$$\begin{cases} x+2y+z=0 \\ y+2z=1 \\ x-3z=-1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -3 & -1 \end{array}\right] \quad R_3 \to R_3 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -1 \end{array}\right]$$

CONTRADICTION

No solution

$$R_3 \to R_3 + 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$0x + 0y + 0z = 1$

(3) Infinite (the solution is a line or plane or somethin not a point)

$$\begin{cases} x+y=1 \\ 2x+2z=4 \\ 2x+y+z=3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 4 \\ 2 & 1 & 1 & 3 \end{array}\right] \quad R_2 \to R_2 - R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{array}\right] \quad R_3 \to R_3 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{-1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivots: 1, −1; basic variables: x, y; free variables: z

(A) $x+y=1$

(B) $-y+z=1$

Let s be a parameter. $z=s$

(B) $\Rightarrow -y+s=1$; $y=s-1$

(A) $\Rightarrow x+y=1$; $x+(s-1)=1$; $x=2-s$

$(x,y,z) = (2-s, s-1, s)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

SOLUTIONS

Unique: No contradictions or free variables

No: a contradiction

Infinite: Free variables

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array} \\
\begin{array}{c} \text{e.g. } A \\ B \\ C \end{array} & \left[\begin{array}{cccccc|c} \boxed{1} & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & \boxed{1} & 0 & -1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 3 \end{array}\right]
\end{array}
$$

already in ref
not in rref

NO contradiction. => At least one sol'n

basic variables: x_1, x_2, x_5

pivots: 1, 1, 1

free variables: x_3, x_4, x_6

Let s, t, u be parameters

$s = x_3$

$t = x_4$

$u = x_6$

(C) $x_5 = 3$

(B) $x_2 - t + 3u = 2$

$x_2 = 2 + t - 3u$

(A) $x_1 + s + t + x_5 + 2u = 1$

$x_1 \quad + 3$

$x_1 = -2 - s - t - 2u$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2-s-t-2u \\ 2+t-3u \\ s \\ t \\ 3 \\ u \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

The sol'n forms a 3-D solid in $\mathbb{R}^6$

a point + 3 directions