Linear Dependence

A set of vectors $\{V_1, V_2 \ldots V_k\}$ is said to be *linearly dependent* if there exists scalars, at least one non-zero, such that

$$S_1 \vec{V_1} + S_2 \vec{V_2} + S_3 \vec{V_3} + \ldots + S_k \vec{V_k} = \vec{0}$$

The set is said to be *linearly independent* if it is not linearly dependent. ie, if and only if, the only solution is S_i are all 0.

Non-trivial. linear relation: at least one non-zero scalar

E.g. 1 I $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.

$$\left.(2)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 3 \end{bmatrix} + (3)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$ a non-trivial relation b/w the 3 vectors

$s(2) + t(4) = 0$; $s = 2,\ t = -1$

$t(3) + u(1) = 0$; $t = -1,\ u = 3$

$2s + 4t = 0$

$3t + u = 0$

E.g. 2 I Is $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ linearly dependent?

Suppose that $s\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2s + t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} (A) \\ (B) \end{matrix}$$

(B) => (A) $2s + (0) = 0$, $s = 0$ Trivial.

No non-trivial relation b/w $[2\ 0]$ and $[1\ 1]$ => linearly independent

E.g. 3 Is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$ linearly dependent?

$$\begin{bmatrix} s + 2t + 4u \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} (A) \\ (B) \end{matrix}$$

$s + 2t = 0$

$s = 2,\ t = -1,\ u = 0$ ← a non-trivial (non all-zero) solution

A non-trivial linear relation b/w the 3 vectors => linearly dependent

Linear Dependence and Independence

A set of vectors $\{v_1, v_2, v_3 .. v_k\}$, $k > 2$ is linearly dependent if and only if, at least one of the vectors can be written as a linear combination of the others.

e.g. $\{[2, 0], [4, 3], [0, 1]\}$ is linearly dependent

$$2[2\ 0] + (-1)[4\ 3] + (3)[0\ 1] = [0\ 0]$$

$$2[2\ 0] = [4\ 3] + (-3)[0\ 1]$$

e.g. $\{[2, 0], [1, 1]\}$ is linearly independent. $[2\ 0]$ is not a linear comb of $[1\ 1]$ & vice versa

(1D) $\{\vec{a}\}$ is linearly independent if, and only if, $a \neq 0$

If $\vec{a} = 0$, it is linearly dependent as multiplying it with any scalar $= \vec{0}$

If $\vec{a} \neq 0$, no non-zero scalar can make it $= \vec{0}$.

(2D) $\{\vec{a}, \vec{b}\}$ is linearly independent if, and only if

① each vector is not a scalar multiple of the other

② area of ▱ spanned by $\vec{a}, \vec{b} \neq 0$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

If the set were linearly dependent, there exists an s and t such that

$s\vec{a} + t\vec{b} = \vec{0}$, and at least one of $\vec{s} \neq 0$, $\vec{t} \neq 0$

$s\vec{a} = -t\vec{b}$

$\frac{-s\vec{a}}{t} = \vec{b}$ $\Rightarrow$ $\vec{b}$ is a scalar multiple of $\vec{a}$.

(3D) $\{\vec{a}, \vec{b}, \vec{c}\}$ is linearly independent if and only if

① none of the vectors can be expressed as a scalar multiple of the others

② they don't lie on the same plane.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

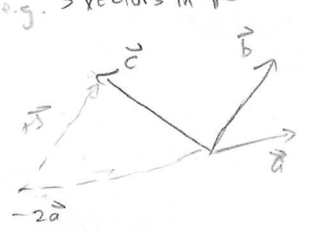
In $\mathbb{R}^2$, $\{\vec{a}, \vec{b}, \vec{c}\}$ must be linearly dependent

Intuition

A set of k vectors is linearly independent, if and only if, their linear combination forms a k-dimensional thing. e.g. $\mathbb{R}^2$-line, $\mathbb{R}^3$-solid

A set of $n+1$ vectors in $\mathbb{R}^n$ must be linearly dependent

e.g. 3 vectors in $\mathbb{R}^2$



$\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

Basis & Base Vectors

A basis of $\mathbb{R}^n$ is a collection of n vectors in $\mathbb{R}^n$ that are linearly independent.

n base vectors form a basis.

Together, they can uniquely describe any vector in $\mathbb{R}^n$ as a linear combination.

Examples:

- $\hat{i}, \hat{j}, \hat{k}$ 
- $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ 2 vectors in $\mathbb{R}^2$
  linearly independent - not scalor multiples of each other.

  $\begin{bmatrix} 100 \\ 200 \end{bmatrix}$ can only be expressed as $100 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

  because $[1,2]$ & $[3,4]$ are not colinear.

Non-examples

- $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$ 3 vectors in $\mathbb{R}^2$ $\ddot{o}$
  automatically linearly dependent
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$ 3 vectors in $\mathbb{R}^3$ ✓
  Linearly dependent

  $a \quad a+b \quad b$

Reasoning: Consider the linear system $\left[\begin{array}{ccccc|c} | & | & | & & | & | \\ v_1 & v_2 & v_3 & \cdots & v_n & w \\ | & | & | & & | & | \end{array}\right] \cdots (*)$

$\{v_1, v_2, v_3 \ldots v_n\}$ lin indept $\Rightarrow$ ref of $(*)$ has $\boxed{\text{no free variables}}$ (i)

$\Rightarrow$ ref of $(*)$ has n basic variables

(n non-zero rows in unagumented matrix)

Since $(*)$ has only n rows $\Rightarrow$ $\boxed{\text{ref of } (*) \text{ has no room for } [0 \; \cdots \; 0 \; | \; \underset{\uparrow \text{ non-zero}}{*}]}$ (ii)

(i) & (ii) $\vec{w}$ can be expressed as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3} \ldots \vec{v_n}$ uniquely.

Determine if the following are linearly independent. Justify.

a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$ No. 4 vectors in $\mathbb{R}^3 \Rightarrow$ linearly dependent

b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ Yes. ① Not scalar multiples of each other

② $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$, Area spanned $\neq 0$

c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ Yes. ① Not scalar multiples of each other

② Non-zero cross product.

Note: the linear combination of the 2 is a plane

d) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ No. $[0\ 0]$ can be expressed as $0\,[1\ 2]$, or w/ any scalar.

If $\vec{0}$ present, linearly dependent

e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$ No ① $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (-1) - 3(-1) + 2(-1)$

$= 0 \Rightarrow$ linearly dependent

② $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

WW3 (2) Is each of the following sets of vectors linearly independent?

a) $\left\{ \begin{bmatrix} -6 \\ 5 \end{bmatrix} \right\}$ Yes, the only s for $s\vec{a}$ that makes $\vec{0}$, is 0.
This is a trivial relation so $\left\{ \begin{bmatrix} -6 \\ 5 \end{bmatrix} \right\}$ is linearly independent

b) $\left\{ \begin{bmatrix} -3 \\ -8 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -12 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 11 \\ 12 \end{bmatrix} \right\}$ No, it is linearly dependent. 4 vectors in $\mathbb{R}^3$, one vector can be described as a linear combination of others

c) $\left\{ \begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix} \right\}$ $\begin{vmatrix} 9 & 2 & 0 \\ -4 & -6 & 0 \\ 5 & 7 & 0 \end{vmatrix} = 0$ Area of parallelopide $= 0$
No, it is linearly dependent.
Area of parallelogram $= 0$

d) $\left\{ \begin{bmatrix} 7 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} \right\}$ $\begin{vmatrix} 7 & -4 & -5 \\ 6 & -1 & 0 \\ 2 & 8 & 3 \end{vmatrix} = 7(-3) + 4(18) - 5(48+2) \neq 0$
Yes, it is linearly independent.

e) $\left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\}$ $\begin{vmatrix} -3 & 0 \\ 2 & 0 \end{vmatrix} = 0$ No, it is linearly dependent.
$[0, 0]$ can be written as $0[-3, 2]$
so that is a linear combination.
If $[0, 0]$ is in the set, linearly dependent

f) $\left\{ \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 8 \end{bmatrix} \right\}$ Yes, cannot be written as scalar multiples of each other
$\Rightarrow$ linearly independent

WW 3 (5) Assume all vectors are non-zero below. Decide whether each statement is true in all cases or false in at least one case

a) If three vectors in $\mathbb{R}^3$ lie in the same plane in $\mathbb{R}^3$, they are linearly dependent. Two linearly independent vectors span a 2-D plane (there is some linear combination of them to map to any point in it), so 3rd linearly dep. True

b) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in $\mathbb{R}^3$ and $\vec{v}_1 = \vec{0}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dep. True.

c) If $\{\vec{u}, \vec{v}\}$, $\{\vec{u}, \vec{w}\}$, and $\{\vec{v}, \vec{w}\}$ are all linearly independent sets, then $\{\vec{u}, \vec{v}, \vec{w}\}$ is also linearly independent. False

d) If $\vec{v}_1$ and $\vec{v}_2$ are non-zero vectors in $\mathbb{R}^3$ and $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. True

e) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ and $\vec{v}_3$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. False, see a.

f) If a set in $\mathbb{R}^3$ is linearly dependent, then the set contains more than 3 vectors False

? g) If S is a set of linearly dependent vectors, then each vector is a linear combination of the other vectors in S False

? h) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a linearly indepent set of vectors in $\mathbb{R}^3$, then $\{v_1, v_2\}$ is also linearly independent True

WW. 3) $\vec{a} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$; $\vec{b} = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix}$; $\vec{c} = \begin{bmatrix} -4 \\ 2 \\ -4 \end{bmatrix}$

Find a non-trivial relation (i.e. not all coefficients are zero) b/w $\vec{a}, \vec{b}, \vec{c}$ if they form a linearly dependent set.

Check for linear dependence:

$$\begin{vmatrix} 4 & -4 & -4 \\ -2 & 2 & 2 \\ 2 & 2 & -4 \end{vmatrix} = 4(-12) + 4(4) - 4(-8) = -48 + 16 + 32 = 0$$

$A_{parallelepipede} = 0$

linearly dependent

Express one as a linear combination of the other

$\vec{a} = s\vec{b} + t\vec{c}$

$2 = -2x - 2y$ (A)

$-1 = x + y$ (B) $\Rightarrow x = -1 - y$

$1 = x - 2y$ (C) $1 = -1 - 3y$

$y = -2/3$

$x = -1 + 2/3$

$x = -1/3$

Check

$2 = -2(-1/3) - 2(-2/3)$ ✓

$-1 = -1/3 + -2/3$ ✓

$1 = -1/3 - 2(-2/3)$

$(x, y) = (-1/3, -2/3)$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = 0$$

$$3\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = 0$$