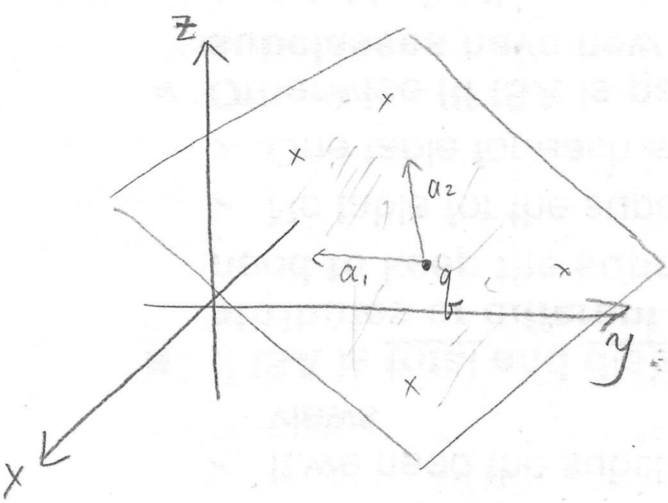
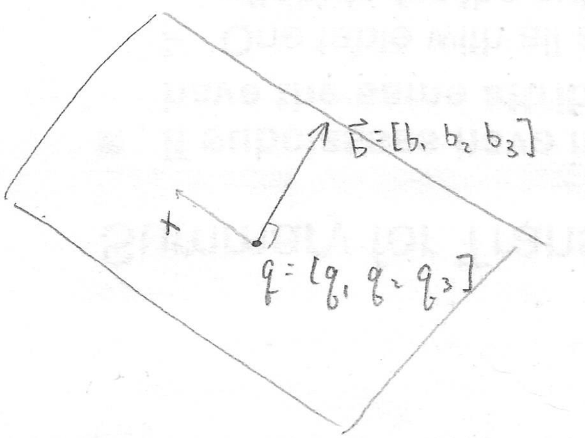
Planes in $\mathbb{R}^3$



Parametric

- a point, $\vec{q}$ on P
- 2 non-collinear directions, $\vec{a_1}$ and $\vec{a_2}$

$$\vec{x} = \vec{q} + s\vec{a_1} + t\vec{a_2}$$



Equation Form

- a point $\vec{q}$ (like a line in $\mathbb{R}_2$)
- a normal direction

$$(\vec{p} - \vec{q}) \cdot \vec{b} = 0$$

$$[x - q_1,\ y - q_2,\ z - q_3] \cdot [b_1,\ b_2,\ b_3] = 0$$

$$b_1(x - q_1) + b_2(y - q_2) + b_3(z - q_3) = 0$$

$$b_1 x + b_2 y + b_3 z = b_1 q_1 + b_2 q_2 + b_3 q_3$$

Eg. Given a plane P and 3 points on it: $A = [0\ 0\ 1]$, $B = [0\ 2\ 0]$, $C = [-1\ 1\ 0]$,
Find the parametric and equation forms of P.

Parametric:

$A = [0, 0, 1]$

$\vec{AB} = B - A = [0, 2, -1]$

$\vec{AC} = C - A = [-1, 1, -1]$

$$[x\ y\ z] = [0, 0, 1] + s[0, 2, -1] + t[-1, 1, -1]$$

Equation:

$A = [0, 0, 1]$ is a point on the plane.

$\vec{AB}$ and $\vec{AC}$ are parallel to the plane,

$\vec{AB} \times \vec{AC} \perp$ to the plane (a normal direction to P)

$$\vec{b} = \vec{AB} \times \vec{AC} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= [-1, 1, 2]$$

$$([x, y, z] - [0, 0, 1]) \cdot \vec{b} = 0$$

$$[x, y, z - 1] \cdot \vec{b} = 0$$

$$(-1)x + (1)y + (2)(z - 1) = 0$$

$$-x + y + 2z = 2$$

Express the plane $x-2y+z=1$ in the following parametric form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} + s\begin{bmatrix} ? \\ 0 \\ ? \end{bmatrix} + t\begin{bmatrix} 0 \\ ? \\ ? \end{bmatrix}$$

coefficients: $[1, -2, 1]$ is $\perp$ to the plane

$s\begin{bmatrix} ? \\ 0 \\ ? \end{bmatrix}$ and $t\begin{bmatrix} 0 \\ ? \\ ? \end{bmatrix}$ are directions of the plane.

$[1, -2, 1] \quad \perp \quad s[?, 0, ?],\ t[0, ?, ?]$

$[1, -2, 1] \cdot [?, 0, ?] = 0$

possibly, $[?, 0, ?] = [1, 0, 1]$

$[1, -2, 1] \cdot [0, ?, ?] = 0$

possibly, $[0, ?, ?] = [0, 1, 2]$

A point on P; satisfy $x-2y+z=1$
$[1, 0, 0]$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

This is one possible parametization

WW 2.17 Write the equation form for the plane defined by

$$x = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

2 directions of the plane

Normal to both (and thus the plane):

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 2 & -3 & 0 \end{vmatrix} = [3, 2, -8]$$

$[3, 2, -8] \cdot [x-5, y, z+3]$

$3x + 2y - 8z = 15 + 24$

$3x + 2y - 8z = 39$

WW 2.13 Find a set of parametric eq'ns that represent the plane $3x_1 + 3x_2 - 3x_3 = 9$

$x_1 = \underline{3} + \underline{9000}\, s$

$x_2 = \underline{-1} + \underline{29}\, t$

$x_3 = \underline{-1} + \underline{9000}\, s + \underline{29}\, t$

A point on P satisfies eq'n: $(3, -1, -1)$

Direction $[a, 0, b]$ is $\perp$ to the normal vector, $[3, 3, -3]$

$[3, 3, -3] \cdot [a, 0, b] = 0$; choose any. $a = 5$, $b = 5$

Direction $[0, c, d]$ is $\perp$ to the normal vector $[3, 3, -3]$

$[3, 3, -3] \cdot [0, c, d] = 0$; choose any. $c = 1$, $d = 1$

We see that $a = b$ and $c = d$, these are directions and so infinite many sol'ns.

WW2.9 Let L be the line containing the point $(1,1,1)$ and perpendicular to the plane $-4x_1 + x_2 + 4x_3 = -4$.

Find a vector eq'n for L. $L: x(t) =$

$-4x + y - 4z = -4$ $\perp$ $(1,1,1)$

$\perp$ $[-4, 1, -4]$ ← direction

$L: x(t) = [1,1,1] + t[-4, 1, -4]$

At what point does L intersect the yz plane?

Set x-coord $= 0$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.25 \\ 0.75 \end{bmatrix}$$

$t = 1/4$

Let $P: \quad x + 2y - z = 1$

$L_1: \quad [0, 2, 4] + s[1, -1, -2]$

$L_2: \quad [0, 2, 4] + t[1, -1, -1]$

Find the intersection points of P & L_1, and P & L_2.

$L_1: [x\ y\ z] = [s, 2-s, 4-2s]$

substitute into p to see if a point exists.

$$s + 2(2-s) - (4-2s) = 1$$
$$s + \cancel{4} \cancel{-2s} \cancel{-4} + \cancel{2s}) = 1$$
$$s = 1$$

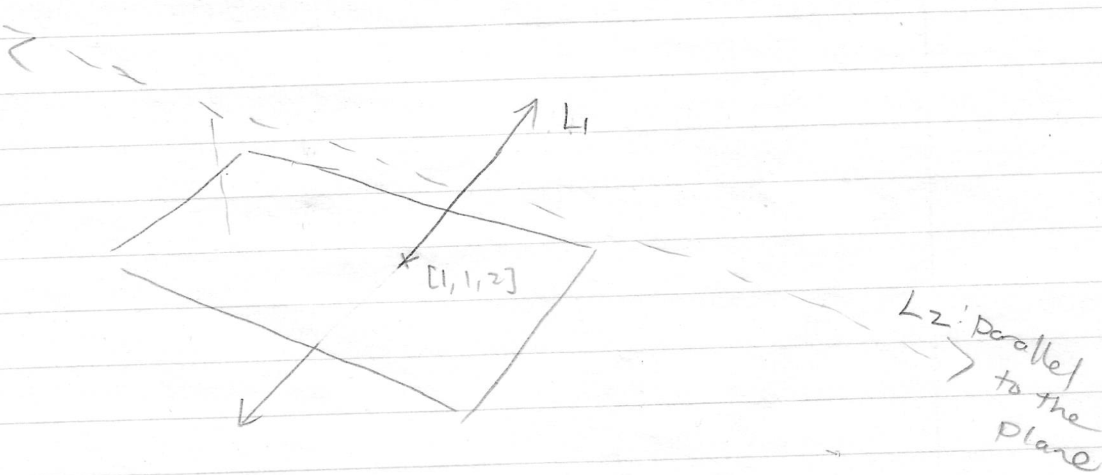
Intersection point: $[0, 2, 4] + (1)[1, -1, -2] = [1, 1, 2]$

$L_2: [x\ y\ z] = [t, 2-t, 4-t]$

substitute into P.

$$t + 2(2-t) - (4-t) = 1$$
$$t + \cancel{4} - 2t \cancel{-4} + t = 1$$
$$0 = 1 \quad \text{contradiction. Impossible.}$$

No t satisfies eq'n. L_2 and P never intersect.



WW 2 (15) Consider the line $L(t) = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ ← a direction

L is ___ (parallel, perpendicular, or neither?) to the plane

a) $3z - (3x + 5y) = -6$

$[-3, -5, 3]$ is normal to the plane

$[-3, -5, 3] \cdot [0, 1, -1] = -8$ Neither

b) $4y - 2x - 3z = 0$

$[-2, 4, -3] \cdot [0, 1, -1] = 1$ Neither

c) $2x - 4y - 4z = 40$

$[2, -4, -4] \cdot [0, 1, -1] = 0$

The normal of this plane is $\perp$ to the line.

The plane and the line are parallel

d) $2z - 2y = 4$

$[0, -2, 2] \cdot [0, 1, -1] =$

This normal is in the (opposite) direction of the direction of the line.

The normal of this plane is parallel to the line.

The plane and the line are perpendicular.

e) $1.5y - 1.5z = 4.5$

$[0, 1.5, -1.5] \cdot [0, 1, -1]$

This normal is in the direction of the line.

The normal of this plane is parallel to the line.

The plane and the line are perpendicular.

WW 2.13 Find an eqn for each of the following planes. Use x, y, z.

a) plane passes through points $(-2,1,4)$, $(-3,3,3)$ and $(1,-1,3)$

2 non colinear directions:

$(-3,3,3)-(-2,1,4) = [-1,2,-1]$

$(1,-1,3)-(-2,1,4) = [3,-2,-1]$

A normal to the plane. cross product of the directions

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -1 \\ 3 & -2 & -1 \end{vmatrix} = [(-2-2), -(1+3), (2-6)] = [-4, -4, -4]$$

$(\vec{q} - [-2,1,4]) \cdot [-4,-4,-4] = 0$

$[x+2, y-1, z-4] \cdot [-4,-4,-4] = 0$

$-4x - 8 - 4y + 4 - 4z + 16 = 0$

$-4x - 4y - 4z = -12 \quad \Rightarrow \quad x + y + z = 3$

b) plane consisting of all points equidistant (equally far) from $(-3,-2,4)$ & $(2,4,1)$

point on the plane: NOT $[(-3,-2,4)-(2,4,1)]\frac{1}{2}$, This is a vector. ($\vec{AC}$)

A × —— × C —— × B

$\vec{AB} = B - A$

$\vec{AC} = \frac{1}{2}(B-A)$ We want a point, C,

$C = A + \vec{AC}$

$= A + \frac{1}{2}(B-A)$

$= \frac{1}{2}(A+B)$

$C = \frac{1}{2}([-1,2,5])$

$= [-\frac{1}{2}, 1, 5/2]$

However, $\vec{AB}$, or $\vec{AC}$ is normal to the plane. $\vec{AB} = [5,6,-3]$

$(\vec{p} - [-\frac{1}{2}, 1, 5/2]) \cdot [5,6,-3] = 0$

$[x+\frac{1}{2}, y-1, z-5/2] \cdot [5,6,-3] = 0$

$5x + 6y - 3z = -5/2 + 6 - 15/2$

c) the plane containing the line $x(t) = \begin{bmatrix} 5 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}$

and perpendicular to the plane $5x + 4y - 2z = 1$

point on the ~~plane~~ line : $[5, 0, -4]$
a direction of the ~~plane~~ line : $[-5, 0, -1]$
P_2 has normal $[5, 4, -2]$, this is a vector on P_1 since $P_1 \perp P_2$

Normal of P_1 : $[-5, 0, -1] \times [5, 4, -2]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 0 & -1 \\ 5 & 4 & -2 \end{vmatrix} = [4, -15, -20]$$

$$(\vec{p} - [5\ 0\ -4]) \cdot [4, -15, -20] = 0$$

$$[x-5, y, z+4] \cdot [4, -15, -20] = 0$$

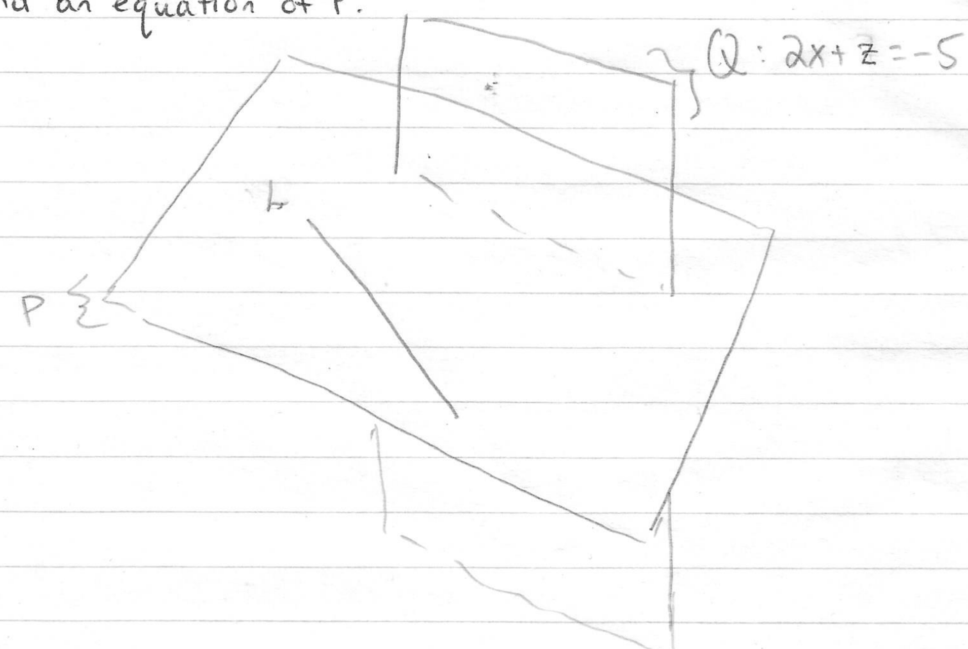
$$4x - 15y - 20z = 20 + 80$$
$$4x - 15y - 20z = 100$$

Let P be a plane in $\mathbb{R}^3$ which contains

$$L(t) = [1, 1, -3] + t[0, 4, 1]$$

and is $\perp$ to the plane $Q: 2x + z = -5$.

Find an equation of P.



① the point $[1, 1, -3]$ is on L, and P,

② $[0, 4, 1]$ is the direction of L, and is thus Parallel to the plane.

③ The plane Q is $\perp$ to $[2, 0, 1]$ (its coefficients), and $Q \perp P$, so P is $\parallel$ to $[2, 0, 1]$

$[2, 0, 1]$ and $[0, 4, 1]$ are both directions of the plane, Cross to get normal direction

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 1 \\ 2 & 0 & 1 \end{vmatrix} = [4, 2, -8]. \text{ This is } \perp \text{ to } P$$

$$[x, y, z] - [1, 1, -3] \cdot [4, 2, -8] = 0$$

$$4(x-1) + 2(y-1) - 8(z+3) = 0$$

Hilroy

$$4x + 2y - 8z = -18$$

WW 2.12 Consider the planes $-x_1 + 2x_2 + 5x_3 = 5$

$-4x_1 + 3x_2 - 5x_3 = 5$

$-x + 2y + 5z = 5$

$-(-4x + 3y - 5z = 5)$

$= 3x - y = 0, \quad y = 3x$

$-x + 2(3x) + 5z = 5$

$5x + 5z = 5$

$z = 1 - x$

a) Find a point P that is on both planes.

$-x + 2(3x) + 5(1-x) = 5$ ✓

$-4x + 3(3x) - 5(1-x) = 5$

$5x - 5 + 5x = 5$

$10x = 10$

$x = 1$

$-4(1) + 3(3(1)) - 5(0) = 5$ ✓

$-1 + 2(3) + 5(0) = 5$ ✓

$(x, y, z) = 1, 3, 0$

$[1, 3, 0]$

b) Find a vector $\vec{v}$ that is parallel to both planes.

$[-1, 2, 5] \perp P_1$

$[-4, 3, -5] \perp P_2$

$[-1, 2, 5] \times [-4, 3, -5]$

orthogonal to both normals:

parallel to both planes

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 5 \\ -4 & 3 & -5 \end{vmatrix} = \left[(-10 - 15), -(5 + 20), (-3 + 8)\right]$$

$$= [-25, -25, 5]$$

c) Find a vector eq'n for the intersection of the two planes

$$x(t) = \underline{[1, 3, 0]} + t \underbrace{[-5, -5, 1]}_{\text{direction}}$$

From b), $[-5, -5, 1]$ is parallel to both planes $\therefore$

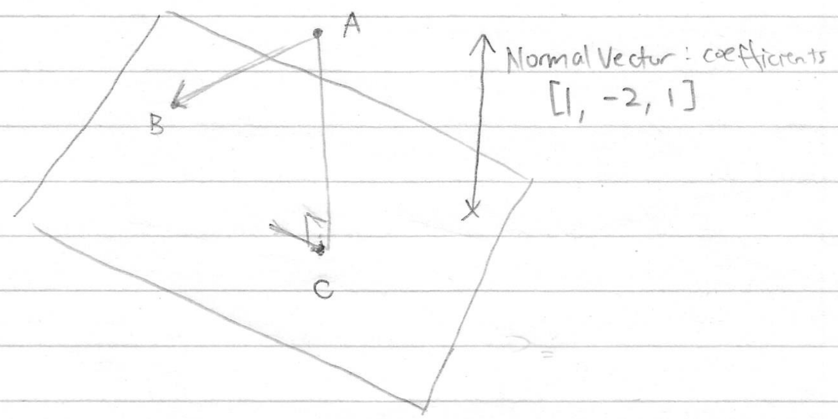
this is the direction of the line

From a), $[1, 3, 0]$ is a point on both planes,

and is a point on the line

Find the distance from A to the plane $P: x - 2y + z = 4$

$A = [2, 1, 1]$



- Let C = the point on the plane closest to A.
- Take any point on the plane. e.g. $[4, 0, 0]$

$\overrightarrow{AC} = \text{proj}_{[1,-2,1]} \overrightarrow{AB}$

$\overrightarrow{AB} = \vec{B} - \vec{A}$

$= [4\ 0\ 0] - [2\ 1\ 1]$

$= [2, -1, -1]$

$= \text{proj}_{[1,-2,1]} [2, -1, -1]$

$= [1, -2, 1] \dfrac{[1,-2,1] \cdot [2,-1,-1]}{[1,-2,1] \cdot [1,-2,1]}$ $\quad \|[1,-2,1]\|^2$

$= \frac{1}{2} [1, -2, 1]$

$\vec{AC} = [\frac{1}{2}, -1, \frac{1}{2}]$

- Distance $= \|\vec{AC}\| = \text{sqrt}(\frac{1}{2}^2 + 1 + \frac{1}{2}^2)$

$= \sqrt{3/2} = \sqrt{6}/2$

Find the coordinate of C.

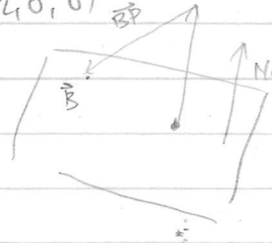
$C = A + \overrightarrow{AC} = [2, 1, 1] + [\frac{1}{2}, -1, \frac{1}{2}]$

$= [2.5, 0, 1.5]$

Hilroy

WW(216) Find the distance from the point $(2,-2,-3)$ to the plane $2x-3y+4z=4$

$B=(2,0,0)$ $\quad P(2,-2,-3)$



Normal $= [2,-3,4]$

$\vec{BP} = [2,-2,-3]-[2,0,0] = [0,-2,-3]$

$\text{proj}_{[2,-3,4]}\overrightarrow{BP}$

$= [2,-3,4]\left(\dfrac{[2,-3,4]\cdot[0,-2,-3]}{29}\right)$

$= [2,-3,4]\left(\dfrac{-6}{29}\right)$

$= \left[\dfrac{-12}{29}, \dfrac{18}{29}, \dfrac{-24}{29}\right]$

$\|\text{proj}_{[2,-3,4]}\vec{BP}\| = \text{sqrt}\left(\dfrac{12}{29}^2 + \dfrac{18}{29}^2 + \dfrac{24}{29}^2\right)$

Describe, using equation and parametric forms,

a) the point $[1, 2]$ in $\mathbb{R}^2$

Parametric:

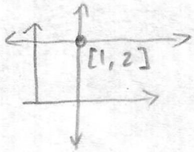
$$[x, y] = [1, 2]$$

Equation:

the intersection of two lines.

$x = 1$

$y = 2$



b) the X-Y plane in $\mathbb{R}^2$

Parametric:

$$[0, 0] + s[1, 0] + t[0, 1]$$

Equation:

No eq'n needed

c) the whole 3-dimensional space in $\mathbb{R}^3$

Parametric:

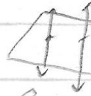
$$[0, 0] + s[1\ 0\ 0] + t[0\ 1\ 0] + u[0\ 0\ 1]$$

Equation:

No eq'n needed

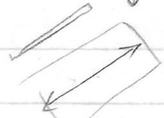
MW 2.11 Determine whether the following statements are True in all cases or false in at least one case. Assume all lines and planes mentioned are in 3D space.

a) Two lines perpendicular to a plane are parallel.



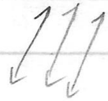
True

b) Two planes parallel to a line are parallel.



False

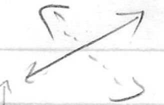
c) Two lines parallel to a third line are parallel.



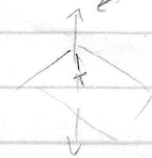
True

d) Two lines either intersect or are parallel

3D!



e) A plane and a line either intersect or are parallel.



True

f) Two planes either intersect or are parallel.

True

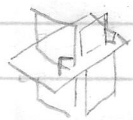
g) Two lines perpendicular to a third line are parallel.

False

h) Two planes parallel to a third plane are parallel.

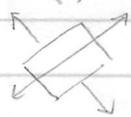
True

i) Two planes perpendicular to a third plane are parallel.



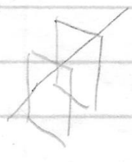
False

j) Two lines parallel to a plane are parallel



False

k) Two planes perpendicular to a line are parallel.



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